A.I. Markushevich

Infinite .....Series

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A. I. Markushevich

Translated and adapted from the third Russian edition (1961) by

IRENA ZYGMUND

SURVEY OF

RECENT EAST EUROPEAN MATHEMATICAL LITERATURE

A project conducted by

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PREFACE TO THE AMERICAN EDITION

INFINITE SERIES is a translation and adaptation of the third Russian edition of a book that was first published thirty years ago. It presents the basic concepts of the theory of infinite series and contains a discussion of the problem of representing elementary functions by means of series without the use of Taylor's formula. The book presupposes a knowledge of high school algebra and trigonometry.

The volume is not intended to serve as a text. Often definitions and theorems are stated in a form that falls short of the standard of rigor customary in modern mathematics. Nevertheless, a genuine effort has been made to give a fundamentally accurate account of the subject. Where steps of a proof using induction are omitted, enough information is supplied so that the reader can complete the proof with relative ease.

The book is intended to help the young reader interested in mathematics to approach infinite series with the facility with which one handles polynomials by following methods begun by Newton and Euler. Of course, infinite series and polynomials are different concepts, and a correct and rigorous treatment of the theory of infinite series must be founded on the theory of limits developed by Cauchy.

This book might be viewed as a mathematical narrative in which the binomial series plays the part of the hero. That many of the proofs and calculations should be lengthy and involved seems unavoidable. Some of these, as well as certain historical remarks, appear in small print. These parts of the text may be omitted on the first reading, although the reader should return to them once he has come to the end of the book.

The student who wishes to learn more about the theory of infinite series may refer to the following books: T. Apostol (Vol. I), *Calculus,* Blaisdell, 1964; T. Apostol, *Mathematical Analysis,* Addison-Wesley, 1957; R. C. Buck, *Advanced Calculus,* Second Edition, McGraw-Hill, 1965; W. Rudin, *Principles of Mathematical Analysis,* Second Edition, McGraw-Hill, 1964; I. I. Hirschman, Jr., *Infinite Series,* Holt, Rinehart and Winston, 1962; K. Knopp, *Infinite Sequences and Series,* Dover Publications, 1956.

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Infinite Series

1. Functions

1. DEFINITION OF A FUNCTION

In the following pages we shall make repeated use of the mathematical concept of a *function.* The mathematical usage of the word "function" differs from its usage in everyday language. For instance, a functional disorder of the heart is some irregularity in the heart's activity. On the other hand, the functions of a person holding public office are his official responsibilities. Thus, when the word "function" is used in everyday language, it normally means either an activity or a responsibility. In both cases, the word is used correctly; moreover, both meanings are in agreement with the meaning of its Latin root word *functio.*

As it is used in mathematics, the word "function" refers to a certain kind of relationship between numerical quantities. For example, it is sometimes said that the area of a circle "is a function of" the length of its radius; that the total pressure exerted by a given mass of gas, at a given temperature, on the walls of its container "is a function of" the volume of the container; that the length of a bar made from a particular kind of metal "is a function of" the temperature of the bar; that the total number of divisors of a given integer "is a function of" that integer; and so on. In none of these examples does the word "function" refer either to an activity or to a responsibility.

Roughly speaking, the mathematical meaning of the word "function" is close to the ordinary meaning of "depends." Thus in each of the above examples, the meaning is not changed too much if one replaces "is a function of" by "depends upon." For example, the statement that the length of a bar made from a particular kind of metal "depends upon" the temperature of the bar has roughly the same meaning as the statement in the third example above.

However, the word "depends" does not convey precisely enough the sense of the modern concept of a function. Suppose that we consider the rye crop in the province of Voronezh. The size of that crop during any year depends mainly on the average rainfall per acre during that year. But a mathematician would not say that the size of

the crop is a function of the average rainfall per acre, because many other factors influence the size of the crop. Again, a person's weight depends upon his height. But a mathematician would not say that the weight of a person is a function of his height.

Looking back over our examples of functional relationships, we see that, in each case, the first quantity mentioned not only depends upon the second, but its value is completely determined by the second. For example, if the length of the radius of a circle is 3 units, then the area of the circle must be $\pi \cdot 3^2 = 9\pi$ square units; a copper bar which is 1 meter long at 0°C. will have a length of 1.000765 meters at 45°C. (since the coefficient of expansion of copper is 0.000017).

However, the size of the rye crop depends not only on the average rainfall per acre, but on a number of other things, such as the quality of the seeds used, the time of year at which the seeds were planted, how well the soil was cultivated. Moreover, so far as we can tell from our present knowledge, the size of the crop can vary from year to year even when these conditions remain unchanged. In this case the value of the second quantity does not determine the value of the first. Similar remarks apply to the example of the relationship between a person's weight and his height.

We are now able to give a definition of the mathematical concept of a function.

DEFINITION. *A quantity* y is a function of *a quantity* x *if to each value of the quantity* x *there corresponds a single definite value of the quantity* y. *The* variable x *is called the* argument *of the function* y.[1]

Note that this definition does not use the ambiguous word "depends." The important thing to remember is that for a functional relationship between two quantities, one single definite value of y is determined once the value of x has been specified.

2. FUNCTIONAL NOTATION

We have a special notation to designate a functional relationship between two quantities. If y "is a function of" x, we write $y = f(x)$. The letter "f" is used because it is the first letter in the Latin root word *functio*. Of course, there is nothing magical about the letter "f". Other letters are often used to denote functions. If r is the radius

[1] A *function* may alternatively be defined as a set of ordered pairs of numbers, no two of which have the same first element.

of a circle, C the circumference of the circle, and A the area of the circle, we could write

$$A = f(r) \quad \text{and} \quad C = F(r). \tag{a}$$

Here we have used different letters, f and F, to denote the functions, because the area and the circumference are two different functions of the radius. If we are given a circle of radius r, we compute its area by squaring r and multiplying by π, whereas to compute the circumference we multiply r by 2π. Thus,

$$A = f(r) = \pi r^2 \quad \text{and} \quad C = F(r) = 2\pi r. \tag{b}$$

These last two formulas contain more information than the two formulas in (*a*). The formulas (*b*) not only tell us that to each definite value of r there correspond definite values of A and C (that is, that A and C are functions of r), but they also give rules for finding the values of A and C that correspond to a given value of r.

3. FUNCTIONS OF SEVERAL VARIABLES

In his study of elementary algebra and geometry the reader has been encountering expressions composed of numbers, letters, symbols for arithmetical operations ($+$, $-$, $\sqrt{\ }$, etc.), symbols for expressing relationships between numerical quantities ($=$, $>$, $\leq$, etc.), parentheses, and so on. Examples of such expressions are:

$$a^3 - 3a^2b + 3ab^2 - b^3, \quad \frac{a^2 - (0.1)b^2}{a^2 + b^2},$$

$$\sqrt[3]{2cx + d}, \quad \frac{2 \sin x + \log x}{\cos^2 x}.$$

(Here we confine our attention to expressions that do not involve symbols for equality or inequality.) In each expression, replacing each letter by a definite number results in an expression that denotes a definite number. (It is understood, of course, that we exclude replacements that would make the denominator of a fraction zero or that would result in a negative number under a radical of the form $\sqrt[2n]{\ }$.) For example, if in the first expression we replace a by 1, and b by 2, we obtain $1 - 6 + 12 - 8 = -1$; if we replace a by 0.5 and b by 0, we obtain $0.125 - 0 + 0 - 0 = 0.125$.

Thus we see that for each of our expressions, replacement of all the variables by particular numbers yields a definite numerical value for the expression in question. This suggests a generalization of our concept of function to include cases in which the values of a quantity are determined by two or more variables. Thus, we shall say that $a^3 - 3a^2b + 3ab^2 - b^3$ "is a function of" a and b, that $\sqrt{2cx + d}$ "is a function of" c, d, and x, and so forth. If we agree to use notation similar to that which we used for functions of a single variable, we can write

$$f(a, b) = a^3 - 3a^2b + 3ab^2 - b^3,$$

$$F(a, b) = \frac{a^2 - (0.1)b^2}{a^3 + b^3},$$

$$\varphi(c, d, x) = \sqrt[3]{2cx + d},$$

$$\Phi(x) = \frac{2 \sin x + \log x}{\cos^2 x}.$$

The small and capital Greek letters φ and Φ ("phi") correspond to the small and capital letters f and F of the English alphabet.

The class of functions of the kind considered above is large, and it is to this class of functions that we shall devote our attention. In the last four examples, there are functions of one, two, and three variables. From this point on we shall be concerned mostly with functions of one variable.

4. FUNCTIONS DEFINED BY FORMULAS

Let us now examine in detail a number of simple examples of functions defined by formulas. The functions we shall consider are defined by

$$y_1 = 1 + x^3,$$
$$y_2 = \sin x,$$
$$y_3 = \log_{10} x.$$

As is always the case when we are dealing with functions, we need only assign a definite numerical value to the argument x in order to

obtain a definite value of the function. The values of these functions are calculated in different ways, however. In the case of the first function, when $x = 0.5$, the corresponding value of the function is

$$y_1 = 1 + (0.5)^3 = 1 + 0.125 = 1.125.$$

We only need a few simple arithmetical operations to calculate the value of this function.

Now consider the second function, defined by

$$y_2 = \sin x.$$

We choose the number $x = \frac{\pi}{9} = 0.34906\ldots$ as a value for the argument. ($\frac{\pi}{9}$ is the radian measure[1] of an angle of 20°.) We cannot immediately calculate the number $\sin \frac{\pi}{9}$ when the only information available is that $\frac{\pi}{9} = 0.34906\ldots$. Usually we get around this difficulty by using a table of trigonometric functions. We find that

$$\sin \frac{\pi}{9} = \sin 20^\circ = 0.342.$$

But this does not solve our problem, since it is still a mystery as to how the table was compiled in the first place. Besides, the number 0.342 is only a good approximation of the value of $\sin \frac{\pi}{9}$, as the reader probably already realizes. Furthermore, if we try to avoid the use of tables by drawing a right triangle with an acute angle of 20° and computing the ratios of measured lengths, the result is not very accurate.

Similar difficulties await us in the case of the function defined by

$$y_3 = \log_{10} x.$$

To illustrate, suppose we set $x = 2$, thereby obtaining $y_3 = \log_{10} 2$. Elementary algebra provides no clue that would enable us to calculate $\log_{10} 2$, and in this case we cannot even resort to a drawing. At

[1] A radian is the measure of a central angle subtended by an arc whose length is equal to the length of the radius of the given circle. In degrees the measure of such an angle is

$$\frac{180^\circ}{\pi} = \frac{180^\circ}{3.141592\ldots} = 57^\circ 17'44.806\ldots''.$$

this stage, we have no choice but to use a table of logarithms[1] to obtain our answer,

$$\log_{10} 2 = 0.3010.$$

Once again we are at a loss when it comes to saying how this answer was obtained by the person who compiled the table.

The examples we have discussed show that there are at least two kinds of functions which can be given by means of formulas. On the one hand, there are those whose values can be calculated from the value of the argument by a succession of arithmetical operations: addition, subtraction, multiplication, and division. Such functions are called *rational* functions. On the other hand, there are those whose values can be computed only if we resort to other devices, such as graphs, geometric constructions, and tables.

[1] We can obtain an approximate value for $\log_{10} 2$ without the use of a table, using the definition of a logarithm. Suppose that we wish an approximation accurate to within 0.1. To do this, we must find two fractions, one lying on either side of $\log_{10} 2$, whose numerators differ by 1 and whose denominators are both equal to 10. Let us denote these fractions by $\frac{p}{10}$ and $\frac{p+1}{10}$, where p is a positive integer. We then have

$$\frac{p}{10} \leq \log_{10} 2 \leq \frac{p+1}{10}.$$

From this it follows that

$$10^{\frac{p}{10}} \leq 10^{\log_{10} 2} \leq 10^{\frac{p+1}{10}}.$$

But by the definition of logarithms, $10^{\log_{10} 2} = 2$; thus we obtain

$$10^{\frac{p}{10}} \leq 2 \leq 10^{\frac{p+1}{10}},$$

or, raising each member of the latter inequality to the tenth power,

$$10^p \leq 2^{10} \leq 10^{p+1}.$$

Hence, to find the desired integer p, we must find two successive powers of 10 such that the number $2^{10} = 2^5 \cdot 2^5 = 32 \cdot 32 = 1024$ lies between them. It is easy to see that $p = 3$, since

$$10^3 < 1024 < 10^4.$$

Thus the desired approximation from below for $\log_{10} 2$ is $\frac{p}{10} = \frac{3}{10} = 0.3$. Similarly, we can obtain an approximation accurate to within 0.0001 by finding the positive integer p such that $10^p \leq 2^{10,000} \leq 10^{p+1}$. To find p, it would be necessary to calculate the ten-thousandth power of 2, or at least to know how many digits it would contain. But this is by no means simple.

5. USES OF INFINITE SERIES IN COMPUTING CERTAIN FUNCTION VALUES

Thus far we have not discussed the extraction of roots. If we are given the function $y = \sqrt{x}$ and wish to compute its value for, say, $x = 10$, we can use a table of square roots or a well-known method for extracting square roots. Unless the number is a perfect square, we obtain only an approximate answer no matter which method we use. In the case of the function $y = \sqrt[3]{x}$, we usually employ a table of cube roots or a table of logarithms. Finally, the only really practical method available for computing values of functions such as $y = x^{\frac{10}{11}} = \sqrt[11]{x^{10}}$ is to use a table of logarithms.

Often we use logarithms or a slide rule to calculate products or quotients, especially in cases where only an approximate answer is required. But how are the tables of logarithms compiled in the first place? And how are the tables of trigonometric functions compiled? The reader will find the answers to these questions in this book.

We shall show how to compute the values of $\sin x$, $\log_{10} x$, etc., by applying arithmetic operations to x. It turns out that no finite number of applications of these operations will give an exact result, although the accuracy of the result increases as we perform more and more operations. Let us clarify this last remark.

Suppose x is the radian measure of a certain angle. Then $\sin x$ can be computed with the help of the following *infinite series:*

$$\sin x = \frac{x}{1} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{x^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \cdots,$$

where the dots indicate that there are an infinite number of terms. Thus the next term in the series would be the fraction

$$- \frac{x^{11}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11},$$

then

$$+ \frac{x^{13}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13},$$

and so on. For each term of the series, there is a term that follows it.

The numerator of the latter term is a power of x greater by two than the power of x in its predecessor; the denominator is a product that contains two more factors than the denominator of the preceding term. Note that for any pair of consecutive terms, one is preceded by a plus sign and the other by a minus sign.

If we discard all the terms of the infinite series after some particular term, the resulting expression is a *polynomial.*[1] For example, if we discard all terms after the second term, we have

$$\frac{x}{1} - \frac{x^3}{1 \cdot 2 \cdot 3} = x - \frac{1}{6}x^3;$$

if we discard all terms after the third term, we have

$$\frac{x}{1} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$$

The higher the degree of the polynomial, that is, the greater the number of terms we retain, the more closely its value approximates the value of $\sin x$. Above, we were looking for the value of

$$\sin \frac{\pi}{9} = \sin 20^\circ = \sin 0.3490658 \ldots,$$

that is, the case in which $x = 0.3490658 \ldots$. Using the polynomials obtained by taking the first one, two, and three terms of the infinite series, we obtain numbers differing by less and less from the actual value of $\sin 20^\circ$:

$$\begin{aligned} x &= 0.3490658 \ldots, \\ x - \frac{x^3}{6} &= 0.3490658 \ldots - \frac{0.0425326 \ldots}{6} = 0.3419770 \ldots, \\ x - \frac{x^3}{6} + \frac{x^5}{120} &= 0.3490658 \ldots - \frac{0.0425326 \ldots}{6} + \frac{0.0051825 \ldots}{120} \\ &= 0.3420202 \ldots. \end{aligned}$$

(It can be shown that the error of the last expression is not more than 0.0000002.) Thus, our infinite series allows us to obtain approximate expressions for the function $\sin x$ in the form of rational functions, in fact, in the form of polynomials. By choosing polynomials of

[1] By definition a polynomial contains a finite number of terms.

sufficiently high degree, we can make the error caused by using them instead of $\sin x$ as small as we like.

Perhaps the reader is under the impression that we are confining our discussion to the trigonometric and logarithmic functions. This is not so; we have a far more ambitious goal in mind. The reader will become acquainted, here, with certain fundamental concepts of higher mathematics; concepts useful not only in mathematics, but also in science and engineering. These concepts are largely the results of the efforts of such great mathematicians as Newton, Euler, Lagrange, Cauchy, and others.

In a book of this size, it is impossible to give a comprehensive account of the theory of infinite series. The book will have served its purpose if it arouses the reader's interest, leads him to further reading, and gives him a deeper understanding of algebra and trigonometry. It is hoped that the reader will be led to reconsider many things which previously seemed too simple, too difficult, or too uninteresting to merit his attention.

2. Newton's Binomial Formula

Alexei Tolstoi once said that in the mind of the general public, logarithms are associated with competitive examinations. This is true, perhaps to an even greater extent, for Newton's binomial theorem. Many high school graduates have a hazy recollection of this theorem as something difficult and required for passing college entrance examinations. Nevertheless, Newton's binomial theorem is one of a few formulas used repeatedly throughout mathematics. In this chapter and in Chapter 5 we shall consider this important formula.

6. MULTIPLYING BINOMIALS—FIRST METHOD

To facilitate our derivation, we first recall that the product of two polynomials is the algebraic sum of all those products of two terms that can be formed by choosing one term from the first polynomial and the other from the second polynomial. Repeated applications of this rule make it possible to write down the product of several polynomials. In particular, this rule can be used to raise binomials (from a late-Latin word meaning "consisting of two terms") to various powers. Thus

$$
\begin{aligned}
(1+x)^2 &= (1+x)\cdot(1+x) = 1\cdot 1 + x\cdot 1 + 1\cdot x + x\cdot x \\
&= 1 + 2x + x^2, \\
(1+x)^3 &= (1+x)\cdot(1+x)\cdot(1+x) = (1+2x+x^2)\cdot(1+x) \\
&= 1\cdot 1 + 2x\cdot 1 + x^2\cdot 1 + 1\cdot x + 2x\cdot x + x^2\cdot x \\
&= 1 + 3x + 3x^2 + x^3, \\
(1+x)^4 &= (1+x)(1+x)(1+x)(1+x) \\
&= (1+3x+3x^2+x^3)(1+x) \\
&= 1\cdot 1 + 3x\cdot 1 + 3x^2\cdot 1 + x^3\cdot 1 \\
&\qquad + 1\cdot x + 3x\cdot x + 3x^2\cdot x + x^3\cdot x \\
&= 1 + 4x + 6x^2 + 4x^3 + x^4,
\end{aligned}
$$

$$
\begin{aligned}
(1 + x)^5 &= (1 + x)(1 + x)(1 + x)(1 + x)(1 + x) \\
&= (1 + 4x + 6x^2 + 4x^3 + x^4)(1 + x) \\
&= 1 \cdot 1 + 4x \cdot 1 + 6x^2 \cdot 1 + 4x^3 \cdot 1 + x^4 \cdot 1 \\
&\qquad + 1 \cdot x + 4x \cdot x + 6x^2 \cdot x + 4x^3 \cdot x + x^4 \cdot x \\
&= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5,
\end{aligned}
$$

and so forth.

Already certain regularities can be observed in the results. First, the degree of the polynomial obtained in each case is equal to the power to which the binomial is raised. In every case, the constant term in the final product is 1, as is also the coefficient of the highest power of x that appears in the product. All the other powers of x between the lowest (zero-th) and highest (equal to the exponent of the binomial) appear with positive coefficients. On the basis of this last statement, we can assert that

$$(1 + x)^6 = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + x^6$$

and

$$(1 + x)^7 = 1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + x^7,$$

where the coefficients a_1, a_2, a_3, a_4, a_5 and $b_1, b_2, b_3, b_4, b_5, b_6$ are certain positive integers. Presently, we shall learn how to compute these coefficients without having to resort to this tiresome multiplication of binomials. We shall derive a formula that will enable us to write any power of a binomial as a polynomial in which the coefficients are simple functions of the exponent.

7. MULTIPLYING BINOMIALS—SECOND METHOD

Before deriving the formula, we shall develop another rule for multiplying polynomials, one that allows us to write the terms of the product of several polynomials directly. Choose a single term from each of the given polynomials and form the product of these terms; the algebraic sum of all possible products of this kind is the product of the given polynomials.

To illustrate, let us use this rule to write the product of the binomials $(5 + x)$, $(1 + 2x)$, and $(3 + x)$. In order to be systematic in our work, we shall determine the partial products so that the final result is a polynomial arranged according to ascending powers of x. Clearly, the constant term in the final product will be the product of the constant terms in the factors:

$$5 \cdot 1 \cdot 3 = 15.$$

The terms involving x arise in several ways. First, we can take the product of the term involving x from the first factor and the constant term from the second and third factors,

$$x \cdot 1 \cdot 3 = 3x.$$

Second, we can take the product of the term involving x from the second factor and the constant terms from the first and third factors,

$$5 \cdot 2x \cdot 3 = 30x.$$

Finally, we take the product of the term involving x from the third factor and the constant terms from the first and second factors,

$$5 \cdot 1 \cdot x = 5x.$$

Similarly, by forming the appropriate products of terms chosen from the factors we can obtain all the terms of the final product that involve x^2. First, we can take the terms involving x from the first and second factors together with the constant term of the third factor and form their product;

$$x \cdot 2x \cdot 3 = 6x^2.$$

Next, we take the terms involving x from the first and third factors along with the constant term from the second factor and form their product;

$$x \cdot 1 \cdot x = x^2.$$

Finally, using the terms involving x from the second and third factors and the constant term from the first factor, we form the product;

$$5 \cdot 2x \cdot x = 10x^2.$$

Last of all, to obtain the term of the final product involving x^3, take the product of the terms involving x in all three factors;

$$x \cdot 2x \cdot x = 2x^3.$$

The product of the three binomials is the algebraic sum of the various products obtained above,

$$\begin{aligned}&(5 + x)(1 + 2x)(3 + x)\\ &= 15 + 3x + 30x + 5x + 6x^2 + x^2 + 10x^2 + 2x^3\\ &= 15 + 38x + 17x^2 + 2x^3.\end{aligned}$$

This method of computing products of polynomials perhaps appears cumbersome and time-consuming, although, once one has practiced with it a bit, products can be written down quite rapidly. At the moment, however, the importance of this rule stems from its usefulness in the derivation of Newton's binomial theorem for positive integral exponents.

8. COMPUTING COEFFICIENT A_{k+1} FROM A_k

We now consider the problem of finding the product of m binomials, each of which is equal to $(1 + x)$; that is, consider the problem of raising the binomial $(1 + x)$ to the mth power. We know that the end result will be a polynomial of the form

$$\begin{aligned}(1 + x)^m = 1 + A_1x + A_2x^2 + A_3x^3 + \cdots\\ + A_kx^k + A_{k+1}x^{k+1} + \cdots + A_{m-1}x^{m-1} + A_mx^m.\end{aligned}$$

We had already noted that the constant term in the result would have to be 1. Indeed, this is clear because the constant term in each of the m binomials

$$\underbrace{(1 + x)(1 + x)\cdots(1 + x)}_{m \text{ factors}}$$

is 1, so that the product of these constant terms is 1.

In the final product, each term involving x is a product of an x-term taken from one of the m factors with the constant terms taken from the remaining $(m - 1)$ factors (all of which are equal to 1). The total number of products that can be formed in this way is the same as the number of ways in which it is possible to select the factor from which the x-term is chosen, namely m. Since each such product will be equal to

$$\underbrace{x \cdot 1 \cdot 1 \cdot \cdots \cdot 1}_{m - 1 \text{ one's}} = x,$$

the x-term in the final product will be mx. Thus, $A_1 = m$.

We could proceed in this way, next finding the value of the coefficient A_2, then that of A_3, and so on. We can save time, however, if instead we can find a general relationship between the coefficients A_k and A_{k+1} of two consecutive terms of the product. Should such a general relationship be discovered, it would be possible to calculate A_2 by making use of the fact that $A_1 = m$; having obtained A_2 it would then be possible to calculate A_3, and so on. Should all of this be possible, there would seem to be every chance that we could discover a general formula for the coefficient A_k in terms of the numbers k and m.

If we wish to find the coefficient of the term involving x^k in the final product, then following the method just presented, we must determine how many products can be formed by choosing the x-term from k of the factors and the constant term from the remaining $m - k$ factors. Figure 1 shows how to obtain all of the products for the x^4-terms in the expansion of $(1 + x)^7$, by selecting four of the seven factors in all possible ways. (There would be ${}_7C_4 = 35$ products in all.) The factors from which the x-term is chosen are circled.

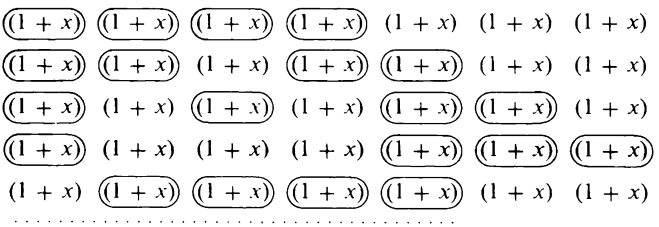


Fig. 1. In each row is indicated a possible selection of factors from which to take the x-terms so as to obtain a product involving x^4 in the expansion of $(1 + x)^7$. For lack of space we have indicated only five of the 35 products involving x^4.

Note that in the general case of $(1 + x)$ to the mth power, the individual products obtained for calculating A_k are all equal to x^k, since in each product of this kind k of the factors are equal to x while the remaining factors are equal to 1. It follows that the number A_k is equal to the number of products that give x^k. Having noted this, we can easily discover the relationship between A_k and A_{k+1}.

To compute A_{k+1}, we must determine in how many ways we can obtain a product equal to x^{k+1} by using m factors, each factor being a term of one of the binomials and no two being from the same binomial. To obtain a product equal to x^{k+1}, we must choose $(k + 1)$ factors to be x-terms of the binomials. The remaining $m - (k + 1)$ choices must be the constant terms of the remaining binomials. Here is a convenient and systematic way to obtain all of these products. Consider the A_k products formed in calculating the coefficient of x^k. For each such product, there are k factors equal to x and $m - k$ factors equal to 1. For a particular product of this kind, consider the binomials from which the x-terms in the product were selected. If we take the same x-terms from these binomials and, in addition, pick an x-term from one of the $(m - k)$ remaining binomials, as well as taking the constant terms 1 from the $m - (k + 1)$ binomials that still remain, we shall have a set of m factors whose product is x^{k+1}. There are $(m - k)$ binomials from which we could choose the $(k + 1)$st x-term, and the new product can be formed by any one of the A_k products equal to x^k. This process is illustrated in Figure 2 for the special case in which $m = 7, k = 4$. The first row indicates a selection of 4 factors from which the x-term is chosen; the fifth x-term, shown in a rectangular box, can be chosen in 3 ways.

$$\begin{array}{ccccccc}
\textcircled{(1+x)} & (1+x) & \textcircled{(1+x)} & (1+x) & \textcircled{(1+x)} & \textcircled{(1+x)} & (1+x) \\
\textcircled{(1+x)} & \boxed{(1+x)} & \textcircled{(1+x)} & (1+x) & \textcircled{(1+x)} & \textcircled{(1+x)} & (1+x) \\
\textcircled{(1+x)} & (1+x) & \textcircled{(1+x)} & \boxed{(1+x)} & \textcircled{(1+x)} & \textcircled{(1+x)} & (1+x) \\
\textcircled{(1+x)} & (1+x) & \textcircled{(1+x)} & (1+x) & \textcircled{(1+x)} & \textcircled{(1+x)} & \boxed{(1+x)}
\end{array}$$

Fig. 2. In the first row is one of the selections of factors from which x-terms can be selected to obtain a product equal to x^4; this selection of factors is the third one of those appearing in Figure 1. In each of the other rows is indicated one of the ways of choosing an additional factor (in a rectangular box) that will give rise to products equal to x^5.

The total number of ways of forming products equal to x^{k+1} in this manner is $A_k \cdot (m - k)$. However, the new selections are *not all distinct*. In other words, if for each of the A_k products equal to x^k we

write down the possible selections of $(k + 1)$ factors, then we shall find repetitions. Consider, for example, the selection in the second row in Figure 2. This is the selection obtained by choosing the x-term from the first, second, third, fifth, and sixth binomials. Figure 2 shows how this is obtained from the first row. But we can start with any one of four other selections of four factors and by adjoining another factor obtain the same selection of five factors. This is clear from Figure 3.

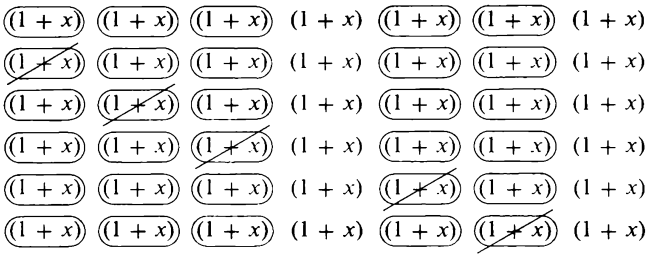


Fig. 3. In the first row we have circled factors from which we take the x-terms in order to obtain a product equal to x^5. Striking out any one of these factors gives a selection for which the corresponding product is x^4.

Thus any one of the $A_k \cdot (m - k)$ new selections can be obtained by adjoining one of the factors in that selection to the remaining factors of that selection. It follows that each selection of $(k + 1)$ factors in the list of

$$A_k \cdot (m - k) \text{ selections}$$

occurs

$$(k + 1) \text{ times.}$$

This means that the total number of *different* (distinct) selections for which the product is x^{k+1} is not $A_k \cdot (m - k)$ but this number divided by $(k + 1)$. Thus, in the product of the m binomials the coefficient A_{k+1} of x^{k+1} is given by the formula

$$A_{k+1} = \frac{A_k \cdot (m - k)}{k + 1}. \tag{1}$$

This gives our relationship between A_k and A_{k+1}.

9. DERIVATION OF NEWTON'S BINOMIAL FORMULA

The relationship which formula (1) of section 8 expresses enables us to calculate the coefficient A_{k+1} once we know the preceding coefficient A_k. Thus, in the expansion of $(1 + x)^m$ the coefficient of x is $A_1 = m$; setting $k = 1$ in formula (1) gives

$$A_2 = A_{1+1} = A_1 \cdot \frac{m - 1}{1 + 1} = A_1 \cdot \frac{(m - 1)}{2} = \frac{m(m - 1)}{2}.$$

Having determined A_2, we can go on to determine A_3; to do this, we set $k = 2$ in formula (1) and obtain

$$A_3 = A_{2+1} = A_2 \cdot \frac{m - 2}{2 + 1} = A_2 \cdot \frac{m - 2}{3}$$
$$= \frac{m(m - 1)}{2} \cdot \frac{m - 2}{3} = \frac{m(m - 1)(m - 2)}{2 \cdot 3}.$$

In general, we can write a formula for any one of the powers of x that appears in the expansion of $(1 + x)^m$. Since we know that

$$A_{k+1} = A_k \cdot \frac{m - k}{k + 1},$$

we see that

$$A_k = A_{k-1} \cdot \frac{m - (k - 1)}{(k - 1) + 1}.$$

Indeed, formula (1) holds for *any* positive integer; therefore, so long as k is an integer greater than 1, formula (1) holds if k is replaced by $(k - 1)$. Furthermore,

$$A_{k-1} = A_{k-2} \cdot \frac{m - (k - 2)}{(k - 2) + 1},$$

$$A_{k-2} = A_{k-3} \cdot \frac{m - (k - 3)}{(k - 3) + 1},$$

$$A_{k-3} = A_{k-4} \cdot \frac{m - (k - 4)}{(k - 4) + 1},$$

Going on to coefficients with smaller and smaller subscripts, we finally arrive at the coefficient A_2 of the x^2-term:

$$A_2 = A_1 \cdot \frac{m - 1}{2}.$$

If we form the product of the expressions on the left-hand sides of these equations and set the result equal to the product of the expressions on the right-hand sides, we obtain

$$A_k \cdot A_{k-1} \cdot A_{k-2} \cdot A_{k-3} \cdot \cdots \cdot A_3 \cdot A_2 = A_{k-1} \cdot \frac{m-(k-1)}{k} \cdot A_{k-2} \cdot \frac{m-(k-2)}{k-1} \cdot A_{k-3} \cdot \frac{m-(k-3)}{k-2} \cdot \cdots \cdot A_2 \cdot \frac{m-2}{3} \cdot A_1 \cdot \frac{m-1}{2}.$$

Notice that A_{k-1} appears as a factor exactly once on both the left and right sides of this last equation. This is also true for each of the numbers A_{k-2}, A_{k-3}, . . . , and A_2. Cancelling equal factors in this equation and using the fact that $A_1 = m$, we obtain

$$A_k = \frac{m-(k-1)}{k} \cdot \frac{m-(k-2)}{k-1} \cdot \frac{m-(k-3)}{k-2} \cdot \cdots \cdot \frac{m-2}{3} \cdot m \cdot \frac{m-1}{2}.$$

This last result can be put in a form that is more convenient and easier to remember by arranging the factors in the denominator in order of increasing magnitude, and by arranging the factors in the numerator in order of decreasing magnitude. Thus we shall have as our final result

$$A_k = \frac{m(m-1)(m-2)\cdots[m-(k-3)][m-(k-2)][m-(k-1)]}{2 \cdot 3 \cdot 4 \cdots (k-2)(k-1)k}. \tag{2}$$

This is the formula that we wished to derive.

In the denominator of the fraction in formula (2) we have the product of all the positive whole numbers that are greater than 1 and less than or equal to k: $2 \cdot 3 \cdot 4 \cdots (k-1)k$. It is customary to write a factor 1 in front of this product (which, of course, does not affect its value). Thus, we have

$$1 \cdot 2 \cdot 3 \cdot 4 \cdots (k-1)k,$$

which is called a *factorial,*[1] or, in this case, *k factorial.* The notation

[1] From the Latin word *factor,* which here means multiplier.

$k!$ is the customary mathematical shorthand for this product; thus:

$$1 \cdot 2 \cdot 3 \cdots (k-1)k = k!$$

For example,

$$\begin{aligned}
1! &= 1, \\
2! &= 1 \cdot 2 = 2, \\
3! &= 1 \cdot 2 \cdot 3 = 6, \\
4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24, \\
5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120, \\
6! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720, \text{ etc.}
\end{aligned}$$

By introducing factorial notation, formula (2) can be written in the form

$$A_k = \frac{m(m-1)(m-2)\cdots(m-k+2)(m-k+1)}{k!}. \quad (2')$$

We can now write Newton's binomial formula in the final form

$$\begin{aligned}
(1+x)^m = 1 &+ \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 \\
&+ \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k \\
&+ \frac{m(m-1)(m-2)\cdots(m-k+1)(m-k)}{(k+1)!}x^{k+1} \\
&+ \cdots + \frac{m(m-1)(m-2)\cdots 3 \cdot 2}{(m-1)!}x^{m-1} \\
&+ \frac{m(m-1)(m-2)\cdots 3 \cdot 2 \cdot 1}{m!}x^m.
\end{aligned} \quad (3)$$

The polynomial on the right in (3) is called the *expansion* of $(1+x)^m$, and the coefficients of the various powers of x are called the *binomial coefficients*. Notice that in this development m and k are both positive integers.

Certain of the coefficients in formula (3) can be simplified. For instance, the product in the numerator of the coefficient of x^m is equal to $m!$, and this coefficient is therefore equal to 1. Replacing $(m-1)!$ by $(m-1)(m-2)\cdots 3 \cdot 2 \cdot 1$ in the denominator of the coefficient of x^{m-1} and cancelling common factors, we obtain m for the coefficient of x^{m-1}, etc.

10. APPLICATIONS

Formula (3) enables us to write formulas for various powers of the binomial $(1 + x)$. Thus, for the special case in which $m = 7$, we have

$$(1 + x)^7 = 1 + \frac{7}{1}x + \frac{7 \cdot 6}{1 \cdot 2}x^2 + \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}x^3 + \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}x^5 + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}x^6 + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}x^7.$$

Reducing the fractions, we obtain

$$(1 + x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7.$$

Although the binomial formula may seem complicated, it is easy to use. It is helpful to remember the following features of the formula: The first term in an expansion is a constant, the second term involves x, the third term involves x^2, and so on, the exponent of x in any particular term after the first being greater by one than the exponent in the preceding term. The last term of the expansion involves x^m. As for the coefficients of the terms in the expansion, the numbers of factors in the numerator and in the denominator each increase by one as we pass from one term to the next. The new factor added in the numerator decreases by one integer as we pass from each term to the next, while the new factor added in the denominator increases by one. If we keep these facts in mind and recall that the coefficient of x is $\frac{m}{1}$, then it is not hard to write out binomial expansions; it is not even necessary to refer to the formula.

To test his skill the reader is urged to write out the expansion of $(1 + x)^{11}$ and to compare his result with ours:

$$(1 + x)^{11} = 1 + \frac{11}{1}x + \frac{11 \cdot 10}{1 \cdot 2}x^2 + \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3}x^3 + \frac{11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}x^5 + \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}x^6 + \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}x^7 + \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}x^8$$

$$+\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}x^9$$

$$+\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}x^{10}$$

$$+\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}x^{11}.$$

Cancelling like factors in the numerators and denominators of the coefficients, we eventually obtain

$$(1+x)^{11} = 1 + 11x + 55x^2 + 165x^3 + 330x^4 + 462x^5 + 462x^6 + 330x^7 + 165x^8 + 55x^9 + 11x^{10} + x^{11}.$$

11. HISTORICAL NOTES

Sir Isaac Newton presented formula (3) of section 9, without proof, in 1676 in his first letter to Oldenburg, secretary of the Royal Society, in London.[1] Newton's second letter to Oldenburg indicates that he had discovered the formula even earlier, about 1664, when he was only twenty-two years old. In this second letter, Newton told how he developed the formula. Simplifying Newton's roundabout procedure, we may say that he considered special cases of powers of the binomial $(1+x)$:

NEWTON (1642–1727)

$$(1+x)^2 = 1 + 2x + x^2,$$
$$(1+x)^3 = 1 + 3x + 3x^2 + x^3,$$
$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4,$$
$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5,$$

and tried to discover a general relation holding between the coefficients of successive terms. He found the relation discussed in section 8,

$$A_{k+1} = A_k \cdot \frac{m-k}{k+1}, \tag{1}$$

[1] Formed in 1645 by a group of scientists.

or, verbally, in the expansion of $(1 + x)^m$, the coefficient in any particular term is equal to the coefficient in the preceding term multiplied by the difference between the power to which the binomial is raised and the power to which x is raised in the preceding term, all this being divided by the power to which x is raised in the term in question. Once this relation has been stated, it is easy to verify for particular powers of the binomial. It is not so simple to discover this relation by piecing together clues gathered from particular examples.

Newton was not the first mathematician concerned with finding the general form of the expansions of binomials raised to various powers. In a work written in 1303 by the Chinese mathematician Chu Shih-Chieh, we find this table of numbers:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

We recognize these numbers as binomial coefficients. The first row corresponds to the zero-th power of $(1 + x)$, since $(1 + x)^0 = 1$; the second row contains the coefficients of the first power of the binomial, $(1 + x)^1 = 1 + x$; the third row contains the coefficients of the second power of the binomial, $(1 + x)^2 = 1 + 2x + x^2$; and so forth, until we come to the ninth and last row, which contains the coefficients for the expansion of $(1 + x)^8$. Chu Shih-Chieh's table ends here. But this brief table enables us to discern a simple rule for writing as many new rows as we please: the sum of any two adjacent numbers in the same row gives the number lying between these two in the next lower row. For example, in the third row take the numbers 1 and 2 (in the order they appear); their sum is 3, which is the number in the fourth row between and beneath 1 and 2. Again, take the two adjacent 3's in the fourth row; their sum is 6, which is the number lying between and beneath them in the fifth row. Moving from left to right along the fifth row, we can obtain the numbers in the sixth row lying between the 1's at the ends of the fifth row: $1 + 4 = 5$, $4 + 6 = 10$, $6 + 4 = 10$, $4 + 1 = 5$.

In general, in the expansion of $(1 + x)^m$, the sum of the coefficients of x^{k-1} and x^k will be equal to the coefficient of x^k in the expansion of $(1 + x)^{m+1}$. In other words, if

$$(1 + x)^m = 1 + A_1x^1 + A_2x^2 + \cdots + A_{k-1}x^{k-1} + A_kx^k + \cdots + A_mx^m$$

and

$$(1 + x)^{m+1} = 1 + B_1x^1 + B_2x^2 + \cdots + B_{k-1}x^{k-1} + B_kx^k + \cdots + B_mx^m + B_{m+1}x^{m+1},$$

then $A_{k-1} + A_k = B_k$ for all $k \leq m$.

To prove this, let us use formula (2′) of section 9 to express A_{k-1} and A_k: We then obtain

$$\begin{aligned} A_{k-1} + A_k &= \frac{m(m-1)\cdots[m-(k-1)+2][m-(k-1)+1]}{1\cdot 2\cdots(k-1)} \\ &\quad + \frac{m(m-1)\cdots(m-k+2)(m-k+1)}{1\cdot 2\cdots(k-1)k} \\ &= \frac{m(m-1)\cdots(m-k+3)(m-k+2)}{1\cdot 2\cdots(k-1)}\left(1 + \frac{m-k+1}{k}\right) \\ &= \frac{m(m-1)\cdots(m-k+3)(m-k+2)}{1\cdot 2\cdots(k-1)} \cdot \frac{m+1}{k} \\ &= \frac{(m+1)m(m-1)\cdots[(m+1)-k+2][(m+1)-k+1]}{1\cdot 2\cdots(k-1)k} \\ &= B_k. \end{aligned}$$

In Chu Shih-Chieh's time formula (2′) of section 9 was unknown and the result concerning sums of successive coefficients could only be approached by examining the particular numbers. It is always possible to extend the table on page 22 to any number of rows by writing 1 as the first number of a new row, then the sums of pairs of adjacent numbers of the preceding row, and, finally, another 1 as the last number in the new row. To illustrate, we copy here the last row given in the table and below it write the next row:

	1	8	28	56	70	56	28	8	1
1	9	36	84	126	126	84	36	9	1

The numbers in this new row are the coefficients for the expansion of $(1 + x)^9$. We suggest that the reader write out the next two rows of the table. The numbers in these two rows are the coefficients for the expansions of $(1 + x)^{10}$ and $(1 + x)^{11}$.

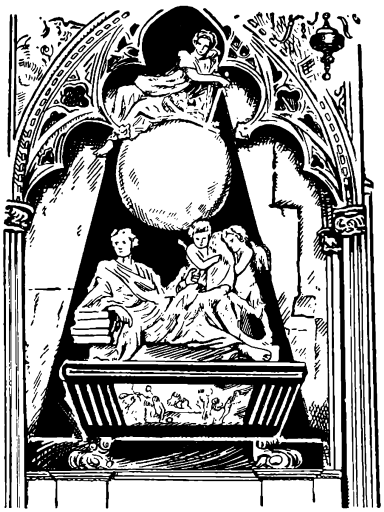
The triangular table, the first lines of which were written down by Chu Shih-Chieh, was also known to certain Indian mathematicians as early as 200 B.C. In the first half of the fifteenth century, mathematicians again studied this triangle. One of them, the mathematician and astronomer Jamshid Ghiyath al-Din al-Kashi of Samarkand, in his work *The Well of Arithmetic* (in Arabic), calls the binomial coefficients of a given row, excluding those at either end (the two 1's), "elements of the indicator of the degree." About them he says, "Know that the element of the indicator of the degree of a square is one number, 2, that for a cube there are two numbers, 3 and 3, that for each (successive) indicator of the degree, their number is increased by one, because of the increase in the number of rows, and that the numbers at the ends are increased accordingly. Adding any two adjacent elements of the indicator of the degree yields the number below these two in the succeeding indicator." As we know, these rules which al-Kashi stated (without proof) enable us to write out the expansion of a binomial for any positive integral exponent. This is the first known formulation of the law of formation of this triangle. Al-Kashi's formulation is, of course, a bit different from Newton's version.

In Europe, this triangle became known through the work *Integral Arithmetic,* published in 1544, by the German mathematician Michael Stifel. In his book, Stifel gave the binomial coefficients for powers up to and including the seventeenth.

Formula (1) of section 8 solves the problem of finding the coefficient of a particular power of x in the expansion of $(1 + x)^m$ for positive integral values of m. This formula was discovered by the English mathematician Briggs (1624), the French mathematician Fermat (1636), and another French mathematician and philosopher, Blaise Pascal (1654), working independently. Pascal wrote a treatise on the triangle of coefficients entitled *Treatise on the Arithmetical Triangle,* which was unpublished until 1665, after his death. It was read by so many people interested in mathematics that the triangle of binomial coefficients acquired the name "Pascal's Triangle." This name is popular today, although it is historically inaccurate. We have seen that this triangle was studied long before Pascal by Chinese, Indian, and Arabian mathematicians.

Perhaps the reader is wondering why the binomial formula given in the last chapter is associated with Newton. It was nothing new to the mathematicians of his day. Newton's fame in this connection

rests on an innovation concerning the permissible values of the exponent m in $(1 + x)^m$. He discovered that the formula could be applied not only for positive integral exponents, but also for *negative* and *fractional* exponents (as will be discussed in succeeding sections), and was the first to use such exponents systematically in algebra. This idea, which had occurred to no other mathematician, was a great breakthrough, and has played a major role in the further development of mathematics. Newton's contemporaries quickly recognized the importance of his discovery. It is said that the binomial formula was engraved on Newton's tombstone, erected at Westminster Abbey shortly after his death.[1]



Newton's Tomb at Westminster Abbey

[1] This story belongs to the eighteenth century, and according to it the formula was engraved above the scroll held by two cherubs standing near the reclining Newton (see the illustration). However, persons who inspected the tombstone late in the nineteenth century reported that they could find no trace of such an inscription. At present, there seems no way to determine whether they failed because the inscription had been obliterated by age, or because there never was such an inscription. Let us take this opportunity to remark that the binomial theorem is not Newton's most outstanding contribution to mathematics. That would undoubtedly be his work in calculus, of which he and his German contemporary Leibniz are generally recognized as co-founders.

12. REVIEW OF RATIONAL EXPONENTS

When we studied elementary algebra, we learned that in order to raise a given number to a positive integral power, such as the mth, we merely find the product of m factors each of which is equal to the given number. The result obtained when we raise a number to a positive integral power is, therefore, a definite integer (positive, negative, or zero). The student will recall that this definition was extended to allow us to speak of powers of numbers with fractional exponents. According to the extended definition, the operation of raising a number to the $\frac{1}{3}$ power is the same as that of taking its principal cube root; raising it to the $\frac{2}{3}$ power corresponds to taking its principal cube root and squaring the result, and so on. In general, if a is a given number and $\frac{p}{q}$ is any fraction with positive integers in both numerator and denominator, then $a^{\frac{p}{q}}$ is defined to be the number obtained by finding the qth root of a and raising the result to the pth power:

$$a^{\frac{p}{q}} = (\sqrt[q]{a})^p.$$

However, this definition, as it stands, may give an ambiguous answer, or even no answer at all. Thus, both $+1$ and -1 will do for the value of 1 raised to the power $\frac{1}{4}$, since

$$(+1)^4 = (-1)^4 = 1.$$

On the other hand, there is no value we can assign to -1 raised to the power $\frac{1}{4}$, since the fourth power of any number is positive. In general, suppose that we wish to raise a number a to power $\frac{p}{q}$. There is no loss of generality if we assume that the fraction $\frac{p}{q}$ is in its lowest terms. In this case we find that difficulties arise only when q is even. Suppose first that a is positive. Then

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

has two possible values, equal in magnitude but opposite in sign. To eliminate this ambiguity, we agree in the future always to mean the *positive* root. Thus, we shall take it that

$$1^{\frac{1}{4}} = \sqrt[4]{1} = 1,$$

$$16^{\frac{1}{2}} = \sqrt{16} = 4,$$

$$4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8, \text{ etc.}$$

Suppose next that a is negative. Then there is no possible value for

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

So in the future when we write expressions of the form $a^{\frac{p}{q}}$, we shall implicitly be excluding this case.

If q is odd, there is no difficulty, since $a^{\frac{p}{q}}$ then has a unique value —positive if a^p is positive, and negative if a^p is negative.

We deal next with the case where the exponent is negative. Suppose that m is any positive rational, and $a \neq 0$. Then if a^m is already defined according to the rules we have given, we define

$$a^{-m} = \frac{1}{a^m}.$$

For example, a^{-2}, a^{-3}, ... mean 1 divided by the square, cube, ... of a. If a^m is not defined, then we do not define a^{-m} either. Also, we do not define 0^{-m}.

Finally, if $a \neq 0$, we define

$$a^0 = 1,$$

and we do not define 0^0.

To sum up:

We have assigned a unique value to every expression of the form a^m for which:

(i) a is real and m is rational (positive, negative, or zero);

(ii) if $a < 0$ and $m = \frac{p}{q}$ in its lowest terms, then q is odd;

(iii) if $a = 0$, then $m > 0$.

13. BINOMIAL FORMULA FOR $m = -1$

Once Newton had convinced himself that formula (3) of section 9 was valid for positive integral exponents, he began investigating the possibility of using this formula for expansions of binomials to fractional and negative powers. This idea seems very natural now. We should remember, however, that Newton was the first to explore this area, and that much of modern algebra is indebted to him, in particular for the systematic use of fractional and negative exponents.

We shall trace Newton's path starting with the simplest case, $m = -1$. The question is, does formula (3) hold when the exponent $m = -1$? Let us observe that proof given in section 6 for formula (3) gives us no reason to assert the validity of the formula for negative exponents. The proof employs the definition of a positive power of a number as the product of a certain number of equal factors and it is a meaningless definition when the exponent is negative or fractional. It is absurd to think that $(1 + x)^{-1}$ is the product of one less than no factors or that $(1 + x)^{\frac{1}{2}}$ is a product of a "half-factor."

We shall make one more important remark. When m is a non-negative integer, then $(1 + x)^m$ is a *polynomial of degree m.* If we compute its coefficients by formula (2′),

$$A_k = \frac{m(m-1)(m-2)\cdots(m-k+2)(m-k+1)}{1\cdot 2\cdot 3\cdots k},$$

we find that A_m, the coefficient of x^m, is equal to

$$\frac{m(m-1)(m-2)\cdots(m-m+2)(m-m+1)}{1\cdot 2\cdot 3\cdots m} = \frac{m(m-1)(m-2)\cdots 2\cdot 1}{1\cdot 2\cdot 3\cdots m} = 1,$$

and all following coefficients

$$A_{m+1}, A_{m+2}, A_{m+3}, \ldots$$

are zero. For example,

$$A_{m+1} = \frac{m(m-1)(m-2)\cdots[m-(m+1)+2][m-(m+1)+1]}{1\cdot 2\cdot 3\cdots m(m+1)} = \frac{m(m-1)(m-2)\cdots 1\cdot 0}{1\cdot 2\cdot 3\cdots m(m+1)} = 0,$$

$$A_{m+2} = \frac{m(m-1)(m-2)\cdots 2\cdot 1\cdot 0\cdot(-1)}{1\cdot 2\cdot 3\cdots m(m+1)(m+2)} = 0,$$

$$A_{m+3} = \frac{m(m-1)(m-2)\cdots 2\cdot 1\cdot 0\cdot(-1)(-2)}{1\cdot 2\cdot 3\cdots m(m+1)(m+2)(m+3)} = 0, \ldots.$$

As we said before, each new factor introduced into the numerator is one less, and into the denominator one more, than its predecessor.

If m is not a positive integer, then $(1+x)^m$ *cannot be a polynomial.* For example, if $(1+x)^{-1} = \frac{1}{1+x}$ were a polynomial of the degree n, then, denoting this polynomial by $P_n(x)$, we would have

$$\frac{1}{1+x} = P_n(x)$$

or

$$1 = (1+x)\cdot P_n(x).$$

But the product $(1+x)\cdot P_n(x)$ would be another polynomial of degree $n+1$ and could not be equal to 1 (a polynomial of degree zero). Therefore, there is no polynomial equal to $(1+x)^{-1}$.

In complete agreement with this, we find that the coefficients A_k computed for m not a nonnegative integer do not vanish. (If all the coefficients following a certain one A were zero, then we would have a polynomial.) For $m = -1$, we obtain the following coefficients:

$$A_1 = \frac{-1}{1} = -1,$$

$$A_2 = \frac{-1(-1-1)}{1\cdot 2} = \frac{(-1)(-2)}{1\cdot 2} = 1,$$

$$A_3 = \frac{-1(-1-1)(-1-2)}{1\cdot 2\cdot 3} = \frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3} = -1,$$

$$A_4 = \frac{-1(-1-1)(-1-2)(-1-3)}{1\cdot 2\cdot 3\cdot 4} = \frac{(-1)(-2)(-3)(-4)}{1\cdot 2\cdot 3\cdot 4} = 1,$$

$$A_5 = \frac{-1(-1-1)(-1-2)(-1-3)(-1-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$

$$= \frac{(-1)(-2)(-3)(-4)(-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5} = -1, \text{ etc.}$$

Therefore, if formula (3) is valid for $m = -1$, it should have the form

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \ldots, \quad (4)$$

where the expression on the right does not have a last term, but every term is followed by another containing x to a higher power. Thus $+x^6$ is followed by $-x^7$, then by $+x^8$, then by $-x^9$, and so on. We cannot call this expression a polynomial, since a polynomial, by definition, contains a finite number of terms, even if very great, and among its terms it is always possible to find a term with x to the highest power.

We could call our expression an "infinite polynomial" but mathematicians use the term *infinite series* (in this case, *power series*). Let us postpone the question of how infinite series are used in mathematics, and begin an examination of formula (4).

14. ALGEBRAIC VERIFICATIONS

First verification. We shall operate with the expression on the right side of formula (4) as if it were an ordinary polynomial, although we know it is not. This dubious procedure was followed by mathematicians of the 17th and 18th centuries. Only later did mathematicians feel a need to exclude concepts and operations that could not be strictly defined. The desire to make mathematics a logically coherent system is completely legitimate and is necessary for the development of modern science. Imposing this discipline upon themselves and their subject has not prevented mathematicians from formulating new concepts. Until a concept has been precisely formulated, we may handle it as freely as did our remote predecessors. But today such free-and-easy treatment rarely goes beyond the walls of the mathematician's "laboratory." Once a work appears in print, everything is so smoothly polished that we have trouble finding the path which led the author to the new concept. But let us return to our subject.

In order to see to what extent (4) can be regarded as an ordinary equation, we might see if the usual rules of algebra apply to it. Let us take the rule that in division, the product of the quotient and the divisor is equal to the dividend. We must check to see if the product

of $(1 - x + x^2 - x^3 + \cdots)$ and $(1 + x)$ is equal to 1. Multiplication gives

$$\begin{array}{l} 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \cdots \\ \underline{1 + x} \\ 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - \cdots \\ \quad x - x^2 + x^3 - x^4 + x^5 - x^6 + x^7 - x^8 + \cdots \\ \hline 1 \end{array}$$

Here, the first and second lines give the multiplicand and multiplier; the third and fourth lines give the partial products. The third line is the partial product obtained by multiplying each term in the series $(1 - x + x^2 - x^3 + \cdots)$ by 1, and the fourth gives the partial product obtained by multiplying each of these terms by x. The fifth line gives the sum of the two partial products. As we can easily see, the terms involving like powers of x cancel, leaving just the number 1 for the product. This verifies Newton's binomial formula for the exponent $m = -1$.

The reader may not be entirely satisfied with this check; our path is slippery. Newton himself was often discontent with only one check, and in many cases (especially for $m = \frac{1}{2}$) resorted to several verifications.

Second verification. We shall now employ a second verification. Applying the familiar rules for dividing one polynomial by another, we obtain

$$\begin{array}{r|l} & 1 - x + x^2 - x^3 + \cdots \\ \hline 1 + x & 1 \\ & \underline{1 + x} \\ & \quad - x \\ & \quad \underline{- x - x^2} \\ & \qquad\quad x^2 \\ & \qquad\quad \underline{x^2 + x^3} \\ & \qquad\qquad - x^3 \\ & \qquad\qquad \underline{- x^3 - x^4} \\ & \qquad\qquad\qquad\quad x^4 \cdots \end{array}$$

Regardless of how far the division is carried, we never obtain a remainder of zero, but keep producing further and further terms of the infinite series (4) obtained above.

Since this second verification may raise more questions than it answers, we shall attempt some arithmetical verifications in the next section.

15. SUBSTITUTING NUMERICAL VALUES

Suppose that in formula (4) of section 13 we put a particular number in place of x.

EXAMPLE 1. Let $x = \frac{1}{4}$ in both $\frac{1}{1+x}$ and $(1 - x + x^2 - x^3 + \cdots)$. The first of these expressions then becomes

$$\frac{1}{1+\frac{1}{4}} = \frac{4}{5} = 0.8,$$

while the second takes the form

$$1 - \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 - \left(\frac{1}{4}\right)^5 + \cdots.$$

Now formula (4) yields

$$0.8 = 1 - \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 - \left(\frac{1}{4}\right)^5 + \cdots,$$

in which "=" must be understood to mean that we cannot merely perform a few of the operations indicated on the right and obtain 0.8. In order to obtain 0.8, we must perform an infinite number of computations. This is a disconcerting task, to say the least. Rather than give up at once, let us continue as best we can. We can begin making the indicated computations, stopping after each term to examine the result. We obtain

$$1 = 1.0,$$

$$1 - \frac{1}{4} = 1 - 0.25 = 0.75,$$

$$1 - \frac{1}{4} + \left(\frac{1}{4}\right)^2 = 1 - 0.25 + 0.0625 = 0.8125,$$

$$1 - \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^3 = 1 - 0.25 + 0.0625 - 0.015625 = 0.796875,$$

$$1 - \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 = 1 - 0.25 + 0.0625 - 0.015625 + 0.00390625 = 0.80078125, \text{ and so forth.}$$

As we expected, none of these intermediate results, 1, 0.75, 0.8125, 0.796875, 0.80078125, . . . , is equal to 0.8. It is clear, however, that by computing the algebraic sum of a larger and larger number of the first terms of the series

$$1 - \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 - \left(\frac{1}{4}\right)^5 + \left(\frac{1}{4}\right)^6 - \cdots,$$

we shall obtain numbers which approach closer and closer to the number,

$$\left(1 + \frac{1}{4}\right)^{-1} = \frac{1}{1 + \frac{1}{4}} = 0.8.$$

In this sense, it is in fact true that the equation

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots \qquad (4)$$

holds for $x = \frac{1}{4}$.

We can carry out a similar check for other values of x. Unfortunately formula (4) fails for some values of x.

EXAMPLE 2. Let $x = -2$ in formula (4). On the left we have

$$\frac{1}{1 - 2} = \frac{1}{-1} = -1,$$

while on the right we obtain

$$1 - (-2) + (-2)^2 - (-2)^3 + (-2)^4 - (-2)^5 + \cdots$$
$$= 1 + 2 + 4 + 8 + 16 + 32 + \cdots.$$

Computing, as before, the sums of one, two, three, four, etc., terms, we obtain

$$1, \quad 1 + 2 = 3, \quad 1 + 2 + 4 = 7, \quad 1 + 2 + 4 + 8 = 15,$$
$$1 + 2 + 4 + 8 + 16 = 31, \ldots$$

and so on. These results not only differ from -1 but also they increase rapidly and show no tendency to approach this value. Therefore, we should admit that formula (4) is not valid for $x = -2$.

16. A FORMULA FOR THE ERROR AFTER n TERMS

The arithmetical check in section 15 showed that Newton's formula (3) for $m = -1$ is valid for some values of x (for example, for $x = \frac{1}{4}$) and false for others (for example, for $x = -2$). So that we may completely answer the question regarding the values of x for which formula (4) holds, we return to the second verification of section 14, where we have the following results:

$$
\begin{array}{r}
1 - x + x^2 - x^3 + x^4 - \cdots \\
1 + x \overline{)\,1} \\
\underline{1 + x} \\
-x \qquad \text{first remainder} \\
\underline{-x - x^2} \\
x^2 \qquad \text{second remainder} \\
\underline{x^2 + x^3} \\
-x^3 \qquad \text{third remainder} \\
\underline{-x^3 - x^4} \\
x^4 \qquad \text{fourth remainder} \\
\underline{x^4 + x^5} \\
-x^5 \qquad \text{fifth remainder}
\end{array}
$$

Let us interrupt the division at some place, for example, at the fifth remainder, and recall that the dividend is equal to the divisor multiplied by the quotient, plus the remainder. We obtain

$$1 = (1 + x)(1 - x + x^2 - x^3 + x^4) - x^5.$$

(The reader can easily verify this equation by multiplying the polynomials on the right and combining like terms.) If we divide each side of the equation by $(1 + x)$, we obtain

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - \frac{x^5}{1 + x}.$$

This result is valuable because it tells us what *error* we would make by limiting the computation of the right side of formula (4) to the first five terms only: $1 - x + x^2 - x^3 + x^4$. This error is clearly

$$-\frac{x^5}{1 + x}.$$

It is obvious that when x is small in absolute value, then x^5 represents a very small fraction and, consequently, the whole error is also

very small. For example, for $x = \frac{1}{4}$ the error is

$$-\frac{\left(\frac{1}{4}\right)^5}{1+\frac{1}{4}} = -\frac{1}{1280} = -0.00078125,$$

which agrees with the result computed on page 32, where the sum of the first five terms for $x = \frac{1}{4}$ was 0.80078125 and therefore the error (the difference between 0.8 and this number) is actually 0.00078125.

On the contrary, if x is large (if the absolute value of x is greater than one), then the absolute value of x^5 is large and, consequently, the error $-\frac{x^5}{1+x}$ is also large. For example, for $x = -2$, we have

$$-\frac{(-2)^5}{1+(-2)} = \frac{32}{-1} = -32,$$

which again agrees with the result on page 33.

It is easy to see that the nth remainder in the division of 1 by $1 + x$ will be $\pm x^n$, where the plus sign is chosen when n is even and the minus sign when n is odd. (In the preceding discussion, we stopped at the fifth remainder, $-x^5$.) We have the following relation between the dividend, divisor, and quotient:

$$1 = (1+x)(1 - x + x^2 - x^3 + x^4 - \cdots \mp x^{n-1}) \pm x^n$$

or, dividing each term of the equation by $1 + x$,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots \mp x^{n-1} \pm \frac{x^n}{1+x}.$$

This equation permits us to evaluate the error made by taking only the sum of the first n terms in the right-hand side of formula (4) and rejecting all the others. This error, equal to

$$\frac{1}{1+x} - (1 - x + x^2 - x^3 + x^4 - \cdots \mp x^{n-1}),$$

is clearly

$$\pm \frac{x^n}{1+x}.$$

17. EVALUATING THE ERROR

The error, $\pm \frac{x^n}{1+x}$, is zero only when $x = 0$. But if the absolute value of x is less than 1 ($|x| < 1$), then the absolute value of x^n is also less than 1. Also, if $|x| < 1$, by making n (the number of terms) sufficiently large, we can make x^n and, hence, $\pm \frac{x^n}{1+x}$ as small in absolute value as we like.

EXAMPLE. If $x = -0.9$, then the error is

$$\pm \frac{(-0.9)^n}{1-0.9} = \pm \frac{(-0.9)^n}{0.1} = \pm 10 \cdot (-0.9)^n.$$

If $n = 1, 2, 3, \ldots$, then the error will be correspondingly 9, 8.1, 7.29, ..., that is, for the present, quite large. For $n = 10$, we have the error equal to $10 \cdot (0.9)^{10} = 3.48674401$; for $n = 20$, the error is $10 \cdot (0.9)^{20} = 1.2157665459\ldots$, still large. However, for $n = 200$, we have $10 \cdot (0.9)^{200} = 0.000000007\ldots$.

In other words, if, wishing to verify the equation

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots$$

for $x = -0.9$, we begin to perform the successive algebraic operations indicated in the right-hand side and take the algebraic sum of the first 20 terms, we obtain a result,

$$8.7842334540\ldots,$$

which differs from the value of the left-hand side

$$(1-0.9)^{-1} = \frac{1}{1-0.9} = \frac{1}{0.1} = 10$$

by more than 1. But if we compute the algebraic sum of the first two hundred terms of the right-hand side, that is, the sum

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + x^{198} - x^{199},$$

then we shall obtain the number

$$9.999999992\ldots,$$

insignificantly different from 10.

18. RANGE OF VALIDITY

We can now assert that the formula (4) holds *for all x for which* $|x| < 1$. That is, if we compute the algebraic sum of the first n terms of the series

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots,$$

then the difference between this sum and $\frac{1}{1+x}$ becomes arbitrarily small as the number n of summands increases without bound. The closer the absolute value of x is to 1, the more terms of the series we have to add in order to obtain an adequate approximation. (This is seen by comparing the examples on page 35 where we performed the computation for $x = \frac{1}{4}$ and on page 36 where $x = -0.9$.)

What can we say when $|x| \geq 1$?

For $x = -1$, we cannot even consider formula (4), since for this value of x the left-hand part, $\frac{1}{1+x}$, of the equation is undefined (the denominator vanishes).

For $x = 1$, we have on the left

$$\frac{1}{1+1} = \frac{1}{2},$$

and on the right

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots.$$

The first term here is one; the sum of the first two is zero; the sum of three is again one, then zero again, and so on. No matter how many terms we add, we always obtain either one or zero, and not only do we never get $\frac{1}{2}$ but we never even approach $\frac{1}{2}$. Therefore, for $x = 1$, formula (4) is not valid.

The expression for the error,

$$\pm \frac{x^n}{1+x},$$

derived on page 35, shows that formula (4) cannot be valid in case $|x| > 1$. In fact, in this case the more terms of the series

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

we add, that is, the larger the n, the greater will be the absolute value

of x^n and, consequently, the greater will be the difference between $\frac{1}{1+x}$ and the sum of the first n terms of the series, since this difference is equal to $\pm \frac{x^n}{1+x}$. We saw this clearly on page 33, assuming $x = -2$.

As a result of this long investigation (which will save us considerable time later), we conclude the following:

The binomial formula (3) *is valid for* $m = -1$ *(then it takes the form* (4)*) only for those* x *such that* $|x| < 1$.[1]

19. BINOMIAL FORMULA FOR $m = \frac{1}{2}$

Newton also checked the binomial formula for other values of m, first for $m = \frac{1}{2}$. At the risk of tiring the reader, we shall repeat Newton's computations. Recall formula (3):

$$(1+x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{m(m-1)(m-2)(m-3)(m-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 + \frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + \cdots.$$

[1] The reader should have been familiar with all this from a course in elementary algebra. In fact, the series $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ represents for $|x| < 1$ an infinite decreasing geometric progression with common ratio $-x$; therefore its sum should be equal to $\frac{1}{1-(-x)} = \frac{1}{1+x} = (1+x)^{-1}$, and we need no further check. We did not refer to the geometric progression solely because we wanted a simple example to prepare the reader for the concept of a series.

The formula for the sum of an infinite geometric progression appeared first in the book *Arithmetical Theory and Practice, Carefully Substantiated* (1656) by the Belgian mathematician André Take. To prove this formula Take used lemmas from a short work *About Measurement of the Parabola* by the Italian mathematician Torricelli, published in 1644. Newton himself used *The Universal Arithmetic* by the English mathematician Wallis (1657) where the formula for the sum of an infinite geometric progression was given.

For $m = \frac{1}{2}$, the left-hand side takes the form

$$(1 + x)^{\frac{1}{2}} = \sqrt{1 + x},$$

and the right

$$1 + \frac{\frac{1}{2}}{1}x + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{1 \cdot 2}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{1 \cdot 2 \cdot 3}x^3$$
$$+ \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} - 3\right)}{1 \cdot 2 \cdot 3 \cdot 4}x^4$$
$$+ \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} - 3\right)\left(\frac{1}{2} - 4\right)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}x^5$$
$$+ \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} - 3\right)\left(\frac{1}{2} - 4\right)\left(\frac{1}{2} - 5\right)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}x^6 + \cdots$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1 \cdot 2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1 \cdot 2 \cdot 3}x^3$$
$$+ \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}x^5$$
$$+ \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}x^6 + \cdots$$

$$= 1 + \frac{1}{2}x + \frac{1}{2 \cdot 4}x^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4$$
$$+ \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}x^5 - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12}x^6 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \cdots.$$

Thus, we have to verify the correctness of the equation

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \cdots, \qquad (5)$$

where the right-hand side is again a power series,[1] but with different coefficients from those in previous examples. To check, we shall square the right-hand side. If formula (5) is correct, then we should get $1 + x$ as the result of squaring.

But to square means to multiply an expression by itself, that is, to perform the multiplication

$$\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \cdots\right)$$
$$\times \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 \cdots\right).$$

We shall do this by writing down terms of the product in rows: in the first row, the product of the multiplicand by the first term of the multiplier, that is, by 1; in the second, the product of the multiplicand by the second term of the multiplier, that is, by $\frac{1}{2}x$; in the third, by $-\frac{1}{8}x^2$; and so on. (We indent each successive row in order to arrange terms containing like powers of x one under another.)

[1] Here it is not immediately clear how, knowing one term, to write the following one. However, this is not difficult to do, using formula (1) of section 8, which can be expressed in words as follows: we have to multiply the coefficient of the given term by the difference between the exponent of the power of the binomial (in this case $\frac{1}{2}$) and the exponent of x in the given term (6) and divide by the exponent of x in the following term (7). Thus, the term $-\frac{21}{1024}x^6$ is followed by the term containing x^7 with the coefficient

$$-\frac{21}{1024} \cdot \frac{\frac{1}{2} - 6}{7} = \frac{33}{2048}.$$

$$
\begin{aligned}
&1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \tfrac{7}{256}x^5 - \tfrac{21}{1024}x^6 + \cdots \\
&\quad + \tfrac{1}{2}x + \tfrac{1}{4}x^2 - \tfrac{1}{16}x^3 + \tfrac{1}{32}x^4 - \tfrac{5}{256}x^5 + \tfrac{7}{512}x^6 - \cdots \\
&\qquad - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 + \tfrac{1}{64}x^4 - \tfrac{1}{128}x^5 + \tfrac{5}{1024}x^6 - \cdots \\
&\qquad\quad + \tfrac{1}{16}x^3 + \tfrac{1}{32}x^4 - \tfrac{1}{128}x^5 + \tfrac{1}{256}x^6 - \cdots \\
&\qquad\qquad - \tfrac{5}{128}x^4 - \tfrac{5}{256}x^5 + \tfrac{5}{1024}x^6 - \cdots \\
&\qquad\qquad\quad + \tfrac{7}{256}x^5 + \tfrac{7}{512}x^6 - \cdots \\
&\qquad\qquad\qquad - \tfrac{21}{1024}x^6 - \cdots
\end{aligned}
$$

Adding the terms in each column, we see that the first two columns give $1 + x$ and that every remaining column gives a zero. For example, in the column containing x^6 the sum of the coefficients is

$$
\begin{aligned}
-\frac{21}{1024} + \frac{7}{512} + \frac{5}{1024} + \frac{1}{256} + \frac{5}{1024} + \frac{7}{512} - \frac{21}{1024} &= \frac{-21 + 14 + 5 + 4 + 5 + 14 - 21}{1024} \\
&= 0.
\end{aligned}
$$

Thus, a formal check verifies equation (5).

A detailed analysis shows, however, that equation (5) does not hold for all values of x but only for those satisfying $|x| \leq 1$. For $|x| > 1$, it fails, because the infinite series on the right does not define a number. Instead of investigating this case further, we shall now get better acquainted with the general concept of a series and its basic properties. After that, we shall be in a position to study Euler's proof of the binomial formula (3) for any exponent m and suitably restricted x.

3. Infinite Series

20. EXAMPLES OF INFINITE SERIES

For purely descriptive purposes, an infinite series may be defined as an algebraic sum in which there are infinitely many summands. Thus every series can be written in the form

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_{n-1} + u_n + u_{n+1} + \cdots,$$

where there is no last term; instead, each term is followed by another one. In Chapter 2, we saw several examples of series:

$$1 - x + x^2 - x^3 + x^4 - \cdots \mp x^{n-1} \pm x^n \mp x^{n+1} \pm \cdots,$$

$$1 - 1 + 1 - 1 + 1 - \cdots \mp 1 \pm 1 \mp 1 \pm 1 \cdots,$$

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \cdots.$$

In the first two examples it is easy, given a particular term, to determine what the next term must be. Thus, in the first example, after x^4 comes $-x^5$, then x^6, and then $-x^7$. In the second example, after 1 follows -1, and then 1 again. In the third example, it is difficult to guess what term follows $-\frac{21}{1024}x^6$. We do not need to be perplexed, however. The rule for writing this series is given in the footnote on page 40.

Mathematicians study not only power series (which could be called "polynomials of an infinite degree") but also other kinds of series, including those whose terms are trigonometric functions. We give one example of a trigonometric series:

$$\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \frac{\sin 9x}{9} + \cdots.$$

21. CONVERGENT AND DIVERGENT SERIES

While we can always add a finite number of terms and get a definite result, adding an infinite number of terms of a series one after another may not give any result. This fact, which we discovered in Chapter 2, deserves further consideration. Essentially, when we are dealing with an infinite number of operations, we cannot expect always to obtain a finite result. Consider the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots .$$

If we start with the first term and add successively the second, the third, and so on, we obtain a sequence of numbers:

$$1, \quad 1 + \frac{1}{2} = \frac{3}{2}, \quad 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}, \quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8},$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{31}{16}, \quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{63}{32},$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{127}{64}, \cdots .$$

None of these numbers will be equal to the sum of all the terms in the series. Looking at this set of separate results, however, we see a remarkable tendency, revealed even more clearly if we write the numbers

$$1, \quad \frac{3}{2}, \quad \frac{7}{4}, \quad \frac{15}{8}, \quad \frac{31}{16}, \quad \frac{63}{32}, \quad \frac{127}{64}, \cdots$$

in the form

$$2 - 1, \quad 2 - \frac{1}{2}, \quad 2 - \frac{1}{4}, \quad 2 - \frac{1}{8},$$

$$2 - \frac{1}{16}, \quad 2 - \frac{1}{32}, \quad 2 - \frac{1}{64}, \cdots .$$

In this way, finding the sums of the first two, three, four, five, . . . , terms of the series (sometimes we call these sums *partial sums* of the series), we discover that the numbers obtained approach closer and closer to a definite number, 2. We call this number 2 the *sum of the series*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots ,$$

and we write

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 2.$$

We must remember that by adding a certain number of terms of a series, starting from the first one (that is, computing a partial sum of the series), we seldom obtain the sum of the series but only approximate values of the sum. The more terms we add the smaller is the error.

The question arises, does every series have a definite sum? In other words, are the partial sums of every series more and more accurate approximations of some number? A simple example shows that the answer is "no." Let us take the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

and start computing its partial sums. Starting with the first term, which is considered as the first partial sum, we obtain the following numbers:

$$1, \quad 1 - 1 = 0, \quad 1 - 1 + 1 = 1, \quad 1 - 1 + 1 - 1 = 0,$$
$$1 - 1 + 1 - 1 + 1 = 1,$$

These numbers,

$$1, \quad 0, \quad 1, \quad 0, \quad 1, \quad 0, \quad 1, \quad \ldots,$$

cannot be considered as more and more accurate approximations of a particular number, and therefore the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

has no sum.

In mathematics, a series that has a definite sum is said to *converge* or to be *convergent;* a series that has no definite sum is said to *diverge* or to be *divergent.* Thus the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

is convergent and its sum is 2; we say that it converges to 2. The series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

is divergent.

22. EARLY DIFFICULTIES WITH DIVERGENT SERIES

During the 18th century the concept of convergence of an infinite series had not yet been formulated with precision. Mathematicians of that day often spoke of the "sum" of a series that we would now classify as divergent. This was because those mathematicians did not define the sum of a series in terms of its partial sums. The considerations that led them to ascribe a sum to a particular divergent series varied from case to case. Usually it was first assumed that a given series had a sum, and then a value was found for it through a more or less complicated chain of reasoning. The following are examples of the methods used to calculate the "sum" of certain divergent series.

If S denotes the "sum" of the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots,$$

then we have

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = S.$$

Transposing the first term on the left to the right side of the equation, we obtain

$$-1 + 1 - 1 + 1 - 1 + \cdots = S - 1.$$

Multiplying each side of the last equation by -1, we have

$$1 - 1 + 1 - 1 + 1 - \cdots = 1 - S.$$

But the left-hand side here is simply S, so that

$$S = 1 - S \qquad \text{or} \qquad 2S = 1,$$

from which we have

$$S = \frac{1}{2}.$$

Therefore (they said) we can write

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}.$$

Other mathematicians arrived at the same result by a different argument. They pointed out that if x is replaced by the number 1 in the series

$$1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \cdots,$$

the result is the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

Since

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \frac{1}{1 + x},$$

it follows that

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{1 + 1} = \frac{1}{2}.$$

There are certain objections to the last argument that compel us to question the validity of the argument and the correctness of the answer to which it led. As we have seen, the series

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

can be obtained from the fraction $\frac{1}{1+x}$ by dividing 1 by $1 + x$. Let us now consider, instead, the fraction $\frac{1}{1+x+x^2}$. Upon dividing 1 by $1 + x + x^2$, we obtain the series

$$1 - x + x^3 - x^4 + x^6 - x^7 + x^9 - x^{10} + \cdots$$

as follows:

$$
\begin{array}{l}
\qquad\qquad 1 - x + x^3 - x^4 + x^6 - x^7 + x^9 - x^{10} + \cdots \\
+x+x^2)\,\overline{1} \\
\quad \underline{1 + x + x^2} \\
\quad\quad - x - x^2 \\
\quad\quad \underline{- x - x^2 - x^3} \\
\quad\quad\quad\quad x^3 \\
\quad\quad\quad\quad \underline{x^3 + x^4 + x^5} \\
\quad\quad\quad\quad\quad - x^4 - x^5 \\
\quad\quad\quad\quad\quad \underline{- x^4 - x^5 - x^6} \\
\quad\quad\quad\quad\quad\quad\quad x^6 \\
\quad\quad\quad\quad\quad\quad\quad \underline{x^6 + x^7 + x^8} \\
\quad\quad\quad\quad\quad\quad\quad\quad - x^7 - x^8 \\
\quad\quad\quad\quad\quad\quad\quad\quad \underline{- x^7 - x^8 - x^9} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad x^9 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{x^9 + x^{10} + x^{11}} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad - x^{10} - x^{11} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{- x^{10} - x^{11} - x^{12}} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad x^{12} \cdots
\end{array}
$$

Using the relation between dividend, divisor, quotient, and remainder, we may establish, by the method used in Chapter 2, that the formula

$$\frac{1}{1+x+x^2} = 1 - x + x^3 - x^4 + x^6 - x^7 + x^9 - x^{10} + \cdots$$

is true for $|x| < 1$. In other words, for any value of x where $|x| < 1$ the partial sums of the series on the right-hand side will tend to the quantity $\frac{1}{1+x+x^2}$ as we continue to include more and more terms to compute the sums. But if we substitute $x = 1$ in the last formula, we obtain

$$\frac{1}{3} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots,$$

that is, we may ascribe to the divergent series under consideration the sum $\frac{1}{3}$ as well as the sum $\frac{1}{2}$.

Similar contradictions often occurred in the works of 18th century mathematicians who tried to clarify them by various methods. Without dwelling on all details of the century-long discussion concerning the summation of series that we now call divergent, we shall present some interesting examples of typical scientific theories of that time.

In one of his works (1710) the Italian Grandi remarked that if in the equation

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2} \qquad (*)$$

we combine the terms by pairs, first with the second, third with the fourth, fifth with the sixth, etc., we obtain

$$0 + 0 + 0 + 0 + 0 + \cdots = \frac{1}{2} \quad \text{or} \quad 0 = \frac{1}{2}.$$

Nowadays we would call this result incorrect and take it as evidence for the necessity of criticizing the method whereby $1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$ was obtained. Grandi, however, did not see matters in this light. Instead, he interpreted his final result $\left(0 = \frac{1}{2}\right)$ as symbolic confirmation that the universe was created out of nothingness.

In another place, Grandi attempts to explain the formula (*) as follows: Let us assume that a father left a precious stone as a legacy to his two sons on condition that the sons take turns keeping the stone for one year each. Then, if we compute the share each brother has in possessing the stone, taking the year of possession as $+1$ and the year when the stone is in the hands of the other brother as -1, we obtain

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

But, on the other hand, according to the conditions in the will the brothers have equal shares of the stone. Consequently, each share is equal to $\frac{1}{2}$. Thus, concludes Grandi, it is clear why

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots = \frac{1}{2}.$$

Regarding this reasoning, the famous philosopher and mathematician Leibniz commented correctly that the infinite series

$$1 - 1 + 1 - 1 + \cdots$$

has no relation to Grandi's example. Indeed, if the brothers possess the stone only two, four, ten, or any even number of years, then their shares would be equal. Then, following Grandi, each brother's inheritance would be $\frac{1}{2}$. At the same time the sum of the first two, four, ten, etc. (any even number will do), terms of the series (*) is zero.

Leibniz attempted to explain the equation (*) in still another way but his explanation is steeped in metaphysics; he refers to the wonderful properties of nature, to the law of justice. These concepts have no place in modern mathematics.

LEIBNIZ (1646–1716)

EULER (1707–1783)

We shall now present still more historical details regarding concepts of convergent and divergent series.

Euler, in the article *About Divergent Series* (1754–1755), calls a series "convergent" if its terms tend to zero and "divergent" otherwise. Ascribing to every series a numerical value, which he calls the "sum" of the series, Euler stresses that the partial sums quite frequently do not appear to be more and more accurate approximations to the sum of the series; on the contrary, the partial sums might differ more and more from the "sum" of the series. Thus, according to Euler, the "sum" of a series is a numerical value of the function whose expansion gives us the series. So, stating that the series $1 - x + x^2 - \cdots$ is the expansion of the function $\frac{1}{1+x}$. Euler ascribes to the series $1 - 1 + 1 - 1 + \cdots$ $(x = 1)$ the "sum" $\frac{1}{1+x} = \frac{1}{2}$; following Euler we would ascribe to the series $1 - 2 + 2^2 - 2^3 + \cdots$ $(x = 2)$ the "sum" $\frac{1}{1+2} = \frac{1}{3}$, etc. In a letter to Goldbach in 1745, Euler describes the complicated computations by which he found that the "sum" of the series $1 - 1! + 2! - 3! + 4! - 5! + \cdots$ equals 0.5963475922. Lagrange also sided to some extent with Euler, ascribing (1770) to every series with terms tending to zero, that is, "convergent" in Euler's sense, a definite sum.

It should be mentioned that some mathematicians proposed definitions of convergence which sound quite modern. In a paper by the English mathematician Waring (1781), we read, "If in $a + b + c + d + e + \cdots$ the sums $a + b$, $a + b + c$, $a + b + c + d$, . . . tend to a finite quantity, which they

approach closer than any given difference, then the series is convergent." It was not this work that determined the development of the theory of series, however. Forty years later, in the preface to his course on *Algebraic Analysis,* Cauchy found it necessary to emphasize, "I say that *a divergent series has no sum,*" and five years later Abel wrote in a letter to his friend Holmboë, "Divergent series as a whole are a fabrication of the devil and it is a disgrace that some people allow themselves to base any proof on them. If we use them, then we may come to whatever conclusion we please, and this results in many difficulties and paradoxes. Can we imagine anything more disgusting than to hear people say that

$$0 = 1 - 2^n + 3^n - 4^n + \cdots,$$

where n is an integer? Is this not really ridiculous?"

"I began to pay a great deal of attention to all this, since, with the exception of the simplest cases, such as geometric progressions, in all mathematics there is hardly a single series whose sum is strictly determined; in other words, this, which is of the greatest importance in mathematics, is without any foundation . . ."

CAUCHY (1789–1859)

ABEL (1802–1829)

These remarks of Cauchy and Abel make it clear that as late as the 1820's the definitions in the theory of infinite series were so diverse and so unsatisfactory that both these mathematicians had to struggle to introduce clarity and order. As a result largely of the critical work of Abel and Cauchy (which provided the foundations for the modern theory of series), divergent series remained out of favor for about 50 years (to the middle of the 19th century). They later reappeared, however, this time on firm foundations and without the metaphysical encumbrances, and they now form a somewhat special subject, closely bound up with their applications. We shall not study such series here, however, since this would require more mathematical sophistication than we can presuppose of the reader.

23. CONVERGENCE OF AN INFINITE SERIES; LIMITS

From this point on, we shall confine our attention to convergent series; we shall assign no numerical sum to any divergent series.

We recall here that an infinite series is an "infinite sum" of the form

$$u_1 + u_2 + u_3 + u_4 + \cdots + u_{n-1} + u_n + u_{n+1} + \cdots ;$$

the series is said to be *convergent* if the partial sums

$$\begin{aligned}
S_1 &= u_1, \\
S_2 &= u_1 + u_2, \\
S_3 &= u_1 + u_2 + u_3, \\
S_4 &= u_1 + u_2 + u_3 + u_4, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
S_n &= u_1 + u_2 + u_3 + u_4 + \cdots + u_{n-1} + u_n, \\
S_{n+1} &= u_1 + u_2 + u_3 + u_4 + \cdots + u_{n-1} + u_n + u_{n+1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

approach closer and closer to some number. More precisely, we define a convergent series as follows:

DEFINITION. *An infinite series is convergent if there exists a number S, such that the sequence of partial sums $S_1, S_2, S_3, \ldots, S_n, \ldots$ approaches arbitrarily close to S: that is, is such that the absolute value of the error in approximating S by S_n can be made as small as desired by choosing n sufficiently large.*

We call the number S the *sum* of the series and we write

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_n + u_{n+1} + \cdots = S.$$

We must now introduce the accepted mathematical terminology for the theory of limits and use it in what follows. Thus, instead of speaking vaguely about the numbers S_n approaching or approximating a number S, we shall say that the numbers S_n *approach the limit* S and we write

$$S_n \to S \quad \text{or} \quad \lim S_n = S,$$

where "lim" is the first three letters of the Latin word *limes* meaning "limit." In addition the notation

$$S_n \to S \quad \text{or} \quad \lim S_n = S$$

will have no other meaning except that the absolute value of the difference $S - S_n$ (the difference between the number S and its approximate value) becomes and stays as small as we please if only the subscript n of the number S_n is sufficiently large (in the case of partial sums, n is the number of added terms of the series). For instance, in case of the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots,$$

the partial sums are

$$S_1 = 1, \qquad S_2 = \frac{3}{2}, \qquad S_3 = \frac{7}{4},$$

$$S_4 = \frac{15}{8}, \qquad S_5 = \frac{31}{16},$$

$$S_6 = \frac{63}{32}, \qquad S_7 = \frac{127}{64}, \cdots,$$

and their limit is the number $S = 2$. Here the differences

$$S - S_1 = 1, \qquad S - S_2 = \frac{1}{2}, \qquad S - S_3 = \frac{1}{4},$$

$$S - S_4 = \frac{1}{8}, \qquad S - S_5 = \frac{1}{16},$$

$$S - S_6 = \frac{1}{32}, \qquad S - S_7 = \frac{1}{64}, \cdots$$

become smaller than any given value, however small. Thus, if we select in advance a very small arbitrary number, for instance 0.000001, then it is always possible to find a number n, starting from which the differences $S - S_n$ will be and remain smaller than 0.000001 in absolute value. It is easy to show that

$$S - S_{21} = \frac{1}{1{,}072{,}576} < \frac{1}{1{,}000{,}000},$$

$$S - S_{22} = \frac{1}{2{,}145{,}152} < \frac{1}{1{,}000{,}000},$$

etc. Thus $S - S_n$ is less than 0.000001 if only $n \geq 21$.

Let us now consider the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\frac{1}{6\cdot 7}+\frac{1}{7\cdot 8}+\frac{1}{8\cdot 9}+\cdots.$$

Its partial sums are

$$S_1=\frac{1}{1\cdot 2}=\frac{1}{2},$$

$$S_2=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3},$$

$$S_3=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{6+2+1}{12}=\frac{3}{4},$$

$$S_4=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$$
$$=\frac{30+10+5+3}{60}=\frac{4}{5},$$

In this example, there is no need to continue the computations, as it is easy to find the rule for forming the partial sums. In order to find it, we shall remark that the terms of the series can be written in the following form: first, $\frac{1}{1\cdot 2}=1-\frac{1}{2}$; second, $\frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3}$; third, $\frac{1}{3\cdot 4}=\frac{1}{3}-\frac{1}{4}$, . . . ; nth term, $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$.

Therefore, a partial sum S_n (that is, the sum of any number n of terms beginning with the first one) is equal to

$$S_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n(n+1)}$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$$
$$+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{1+n}\right)$$

$$=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{n-1}-\frac{1}{n}+\frac{1}{n}-\frac{1}{n+1}$$

$$=1-\frac{1}{n+1}.$$

It is clear that when n increases indefinitely the number $\frac{1}{n+1}$

will become arbitrarily close to zero and the partial sum S_n will approach 1. In other words,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.^1$$

It follows from this that the given series is convergent and that its sum is one.

Returning to the general case, we stress that the essence of approaching a limit lies in the possibility of making the quantity $|S - S_n|$ *smaller than any given positive number, no matter how small, by taking the number n sufficiently large.*

The concept of a limit is important not only in the theory of infinite series, but in all of mathematics. The reader has seen it, for instance, in geometry, where the length of the circumference of a circle was defined as the limit of the perimeters of inscribed and circumscribed regular polygons, and the volume of a cylinder and a cone as limits of volumes of inscribed or circumscribed prisms and pyramids, etc. In a discussion of limits, it is helpful to use the geometric representation of numbers on a number line (Fig. 4):

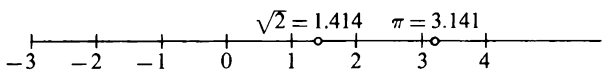


Fig. 4.

On a straight line, let us fix one point as the starting point and let it represent 0, and then establish a unit of length and lay off segments of various lengths to the right and to the left of the point representing zero. Thus we can represent numbers geometrically. To every positive number, we shall ascribe a point on our line to the right of the point representing zero and such that the distance between this point and the point for 0, measured in the established units, will be equal to the given number. Analogously, to each negative number will correspond a definite point on the line to the left of the point for zero. Thus to every positive or negative number corresponds a point on the line. Conversely, to every point on the line corresponds a definite number. It is positive if the point is to the right of the point zero and negative if it is to the left of the point zero. We shall call this line a *number line.* For brevity, instead of saying, "the point representing the number +2," "the point representing the number zero," etc., we shall say, "the point two," "the point zero," etc.

[1] We write $n \to \infty$ (∞ is the symbol for infinity) to show that n increases without bound.

On a number line we can show how a sequence of numbers approaches a limit. Thus, plotting successively the points

$$S_1 = 1,$$

$$S_2 = \frac{3}{2},$$

$$S_3 = \frac{7}{4},$$

$$S_4 = \frac{15}{8},$$

$$S_5 = \frac{31}{16},$$

$$\ldots,$$

we see clearly that these points approach the point $S = 2$, clustering up to it (Fig. 5).

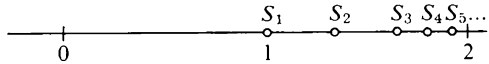


Fig. 5

It is true that Figure 5 would soon cease to help us, since the points S_6, S_7, . . . nearly coincide with the point S.

In general, the tending of any numbers $S_1, S_2, S_3, S_4, \ldots, S_n, \ldots$ to the limit S is expressed graphically by the clustering around a limit point of points corresponding to the later numbers in the sequence. It is important to observe the basic characteristics of this clustering. If we try to "separate" the point S from the points S_n, which approach it as a limit, by means of a very small interval containing the point S as the midpoint, we shall not succeed. In fact, outside such an interval, or we could say, "beyond the fence," there will always be only a finite number of points. The whole infinite collection of the remaining points will be within the interval, no matter how small it is. In other words, we may take an interval with the center at the point S so small that outside of it there are hundreds, thousands, millions of the points S_n. But the bulk, the countless "majority" of these points will still be squeezed within the cramped space of that interval—around the point S.

24. CAUCHY'S CRITERION

We shall use the geometric representation of numbers of section 23 in order to make clearer the *basic principle (criterion) of convergence of infinite series,* first formulated by Cauchy (page 49).

When it is necessary to prove that a definite number S is the sum of an infinite series

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_n + u_{n+1} + \cdots,$$

then it is possible simply to compute the partial sums

$$S_1 = u_1, \qquad S_2 = u_1 + u_2, \qquad S_3 = u_1 + u_2 + u_3, \quad \ldots,$$

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n, \quad \ldots,$$

and see if they approach the specified number S. In other words, we can compute the absolute values of the differences

$$|S - S_1|, \quad |S - S_2|, \quad \ldots, \quad |S - S_n|, \quad |S - S_{n+1}|, \quad \ldots,$$

and try to establish that these numbers, from some point on, are smaller than any positive number given in advance. For example, we can do this if we wish to verify the validity of the binomial formula

$$(1 + x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$$

for any particular value of x, let us say for $x = 0.21$. Then on the left-hand side we shall have

$$(1 + 0.21)^{\frac{1}{2}} = \sqrt{1.21} = 1.1,$$

and we have only to verify that the partial sums of our infinite series tend to 1.1.

The sum S of an infinite series is not always known beforehand, however. More often we have to determine whether it exists, that is, whether the given infinite series is convergent and, if it is, what its sum is. In such cases Cauchy's criterion is extremely important. This is how it works.

We assume first that the series is convergent. Then its partial sums

$$S_1, S_2, S_3, S_4, \ldots, S_n, S_{n+1}, \ldots$$

approach a definite limit S. If we plot the point S and the points S_1, S_2, S_3, S_4, . . . , S_n, . . . on a number line, then the S_n will cluster

around S. Indeed, we can take an arbitrarily small interval with S as its midpoint (Fig. 6), and all the points representing the partial sums,

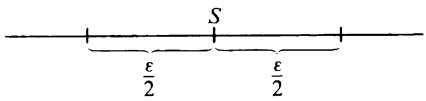


Fig. 6

starting from some certain one, will lie within this interval. Let ε denote its length and assume that, starting from a certain subscript n, the point S_n and the following points S_{n+1}, S_{n+2}, S_{n+3}, . . . , S_{n+p}, S_{n+p+1}, . . . all lie within this interval. It is easy to see that the distance between S_n and any one of the points S_{n+1}, S_{n+2}, S_{n+3}, . . . , S_{n+p}, . . . is smaller than ε. But these distances are clearly

$$|S_n - S_{n+1}|, |S_n - S_{n+2}|, \ldots, |S_n - S_{n+p}|, |S_n - S_{n+p+1}|, \ldots.$$

Thus,

$$|S_n - S_{n+p}| < \varepsilon,$$

where ε is an arbitrarily small positive number, n is a sufficiently large number, and $(n + p)$ is any number greater than n. Thus, we have:

CAUCHY'S CRITERION (necessary condition)[1]. *When an infinite series is convergent, the absolute value of the difference between two partial sums becomes and stays smaller than an arbitrarily small number ε given beforehand:*

$$|S_n - S_{n+p}| < \varepsilon,$$

for all sufficiently large n and any positive p.

Roughly speaking, Cauchy's criterion states that more and more accurate approximations of a quantity should differ less and less from each other. The following important corollary results:

COROLLARY. *If a series*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is convergent, its terms approach zero as n increases; that is, $u_n \to 0$ when $n \to \infty$.

Proof. In a convergent series, according to Cauchy's criterion, the quantity $|S_{n+p} - S_n|$ approaches zero when n increases without bound. Since it is valid when p is any positive integer, it is valid for

[1] Sufficiency is discussed on page 58.

$p = 1$. Then we have

$$S_{n+1} - S = (u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1}) - (u_1 + u_2 + \cdots + u_n) = u_{n+1}.$$

Thus the quantity

$$S_{n+1} - S_n = u_{n+1}$$

should approach zero when n approaches infinity.[1]

For this reason, we may be certain that the series

$$\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \sin 4^\circ + \sin 5^\circ + \cdots + \sin n^\circ + \cdots$$

is divergent. Indeed, among its terms we have $\sin 90^\circ = 1$, and $\sin 270^\circ = -1$, $\sin 450^\circ = 1$, $\sin 630^\circ = -1, \ldots$, and additional terms whose absolute value is equal to one will appear no matter how far we move away from the beginning of the series (for example, in the 360,000,090th place we shall have the term $\sin 360000090^\circ = \sin (1{,}000{,}000 \cdot 360^\circ + 90^\circ) = \sin 90^\circ = 1$). Thus, the terms of the series do not approach zero and the series cannot converge. (Should it converge, by Cauchy's criterion its terms would approach zero.)

It also follows from Cauchy's criterion that *the sum of any number of successive terms of a convergent series tends to zero as we take the first term of this sum farther and farther away from the first term of the series.* To prove this, remember that Cauchy's criterion asserts that in a convergent series

$$S_{n+p} - S_n \to 0 \quad \text{when } n \to \infty.$$

Here, p is a positive integer which need not be constant and which we may, if we like, take to be a function of n. Now

$$\begin{aligned} S_{n+p} - S_n &= (u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + u_{n+2} + \cdots \\ &\quad + u_{n+p}) - (u_1 + u_2 + u_3 + \cdots + u_n) \\ &= u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p}, \end{aligned}$$

and therefore for a convergent series the sum

$$u_{n+1} + u_{n+2} + \cdots + u_{n+p} \to 0 \quad \text{when } n \to \infty.$$

Thus, for instance, the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \cdots,$$

in which the terms approach zero as they move away from the begin-

[1] Clearly, this result ($u_{n+1} \to 0$ when $n \to \infty$) is the same as the initially formulated one ($u_n \to 0$ when $n \to \infty$), since they both mean that the magnitude of a term of a convergent series tends to zero as the index of the term increases indefinitely.

ning of the series, is divergent since the sum of all identical terms is always equal to one, that is, does not approach zero.

Let us consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots,$$

which is called the *harmonic series.* Its terms again approach zero. Nevertheless, this series diverges, since the sum of its terms from the term $\frac{1}{2^n + 1}$ to the term $\frac{1}{2^{n+1}}$ $\left(= \frac{1}{2^n + 2^n}\right)$ equals

$$\underbrace{\frac{1}{2^n + 1} + \frac{1}{2^n + 2} + \frac{1}{2^n + 3} + \cdots + \frac{1}{2^n + 2^n}}_{2^n \text{ terms}}$$

$$> \underbrace{\frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}}}_{2^n \text{ terms}} = \frac{2^n}{2^{n+1}} = \frac{1}{2},$$

and, consequently, does not approach zero.

Cauchy not only gave a *necessary* condition for the convergence of a series (that is, a condition the lack of which makes a series diverge) but also a *sufficient* condition for the convergence of a series:

CAUCHY'S CRITERION (sufficient condition): *If in some series*

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_n + u_{n+1} + \cdots$$

the absolute value of the difference between two partial sums,

$$|S_{n+p} - S_n|,$$

that is, the absolute value of

$$u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p}$$

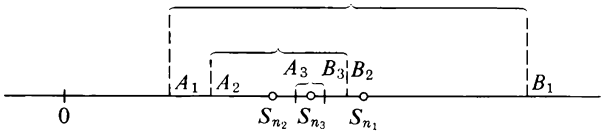
for any p can be made arbitrarily small for all *n larger than a suitably chosen value, and for* all *p, then the series converges.*

Proof. To see why this is so, we shall investigate how the points representing the partial sums are distributed on the number line.

Let us assume that the numbers $|S_{n+p} - S_n|$ can be made as small as desired by choosing n large enough. Then we can select a number n_1 sufficiently large so that all partial sums following S_{n_1} will differ from it by no more than 0.1, for instance. Laying off the length 0.2 from the point S_{n_1} in both directions, we obtain the interval A_1B_1

(Fig. 7) having the combined length 0.4; all the points representing S_{n_1+1}, S_{n_1+2}, S_{n_1+3}, S_{n_1+4}, . . . will lie in this interval:

Fig. 7



Also, since the length of the interval is 0.4 and S_{n_1+1}, S_{n_1+2}, S_{n_1+3}, . . . differ from S_{n_1} (which is the midpoint of the interval) by no more than 0.1, the points S_{n_1+1}, S_{n_1+2}, S_{n_1+3}, . . . will not approach the end points A_1 and B_1 of the interval any closer than 0.1. Now let us take another partial sum S_{n_2}, with such a large subscript $n_2 (n_2 > n_1)$ that all the following sums will differ from it by less than 0.01, and let us lay off the length 0.02 to the right and to the left of the point S_{n_2}. We then obtain a new interval A_2B_2, contained entirely in A_1B_1. In addition, all the points S_{n_2+1}, S_{n_2+2}, S_{n_2+3}, . . . will lie in this interval closer to its midpoint than to its ends and approach the end points A_2 and B_2 no closer than 0.01.

Then we select a partial sum S_{n_3} with a still larger subscript n_3 $(n_3 > n_2 > n_1)$, so that all following partial sums S_{n_3+1}, S_{n_3+2}, S_{n_3+3}, . . . will differ from S_{n_3} by less than 0.001. Laying off the length 0.002 from the point S_{n_3} in both directions, we obtain a very small interval A_3B_3 inside which (at a distance of at least 0.001 from its end points) will lie all the points S_{n_3+1}, S_{n_3+2}, S_{n_3+3}, . . . representing partial sums which follow S_{n_3}.

Continuing with this procedure, that is, finding point S_{n_4} $(n_4 > n_3 > n_2 > n_1)$ such that $|S_{n_4+p} - S_{n_4}| < 0.0001$, then point $S_{n_5}(n_5 > n_4 > n_3 > n_2 > n_1)$ such that $|S_{n_5+p} - S_{n_5}| < 0.00001$, and so on, we obtain a set of intervals A_1B_1, A_2B_2, A_3B_3, A_4B_4, A_5B_5, A_6B_6 . . . of smaller and smaller length, 0.4, 0.04, 0.004, 0.0004, . . . , etc., each containing the next. In addition, each of these intervals will contain all partial sums starting from a certain one (namely, A_1B_1 will contain all starting with S_{n_1}, A_2B_2—all starting with S_{n_2}, A_3B_3—all starting with S_{n_3}, and so on.) Clearly, all these intervals will close in on some single point S lying inside each of them. It is also clear that the points representing partial sums must condense around just this point. Therefore, S is the limit of the numbers S_1, S_2, S_{n_1}, S_{n_1+1}, . . . , S_{n_2}, . . . , S_{n_3}, . . . , which can be considered as more and more accurate approximations of S. We have proved that $S_n \to S$ when $n \to \infty$, which means that the series from which we were taking the partial sums, is convergent and its sum equals S.

EXAMPLE. To illustrate how Cauchy's criterion can be applied, we shall show that the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots$$

is convergent. We must consider the sum of an arbitrary number of its terms starting from the $(n+1)$st term,

$$\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \cdots + \frac{1}{(n+p)^2},$$

and prove that this sum can be made arbitrarily small by making n sufficiently large (regardless of the value of p). The proof is almost immediate if we notice that

$$\begin{aligned}
&\frac{1}{(n+1)^2} < \frac{1}{n(n+1)},\\
&\frac{1}{(n+2)^2} < \frac{1}{(n+1)(n+2)},\\
&\frac{1}{(n+3)^2} < \frac{1}{(n+2)(n+3)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{1}{(n+p)^2} < \frac{1}{(n+p-1)(n+p)}.
\end{aligned}$$

(The fractions on the right-hand side of each of the above inequalities have smaller denominators than those of the fractions on the left and the fractions on the right are therefore larger.) Also, we can express the fractions on the right-hand side as differences between two fractions as follows:

$$\begin{aligned}
\frac{1}{n(n+1)} &= \frac{1}{n} - \frac{1}{n+1},\\
\frac{1}{(n+1)(n+2)} &= \frac{1}{n+1} - \frac{1}{n+2},\\
\frac{1}{(n+2)(n+3)} &= \frac{1}{n+2} - \frac{1}{n+3},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{1}{(n+p-1)(n+p)} &= \frac{1}{n+p-1} - \frac{1}{n+p}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\cdots+\frac{1}{(n+p)^2}\\
&\quad<\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\cdots+\frac{1}{(n+p-1)(n+p)}\\
&\qquad=\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+2}-\cdots\\
&\qquad\qquad+\frac{1}{n+p-1}-\frac{1}{n+p}=\frac{1}{n}-\frac{1}{n+p}<\frac{1}{n}.
\end{aligned}$$

Thus,

$$\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\cdots+\frac{1}{(n+p)^2}<\frac{1}{n},$$

that is, this sum can be made arbitrarily small by making n sufficiently large, whatever the value of p. Cauchy's criterion now tells us that the given series is convergent.

4. Tests for the Convergence of Series

In practice, in order to show that some particular series is or is not convergent, we seldom use Cauchy's criterion, but instead we use various special criteria. Here we shall consider only two of these criteria: Leibniz's test for alternating series with decreasing terms and d'Alembert's ratio test for series with positive terms.

25. ALTERNATING SERIES

DEFINITION. *An infinite series is said to be an* alternating series *if its terms are alternately positive and negative.*

The following are examples of alternating series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots, \qquad (\alpha)$$

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{7}} - \cdots, \qquad (\beta)$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \frac{1}{2^4} + \frac{1}{9} - \frac{1}{2^5} + \frac{1}{11} - \cdots, \qquad (\gamma)$$

$$\frac{\sin^2 1^\circ}{1} - \frac{\sin^2 2^\circ}{2^2} + \frac{\sin^2 3^\circ}{3^2} - \frac{\sin^2 4^\circ}{4^2} + \frac{\sin^2 5^\circ}{5^2} - \cdots, \qquad (\delta)$$

$$2 - 1\frac{1}{2} + 1\frac{1}{4} - 1\frac{1}{8} + 1\frac{1}{16} - 1\frac{1}{32} + 1\frac{1}{64} - \cdots. \qquad (\varepsilon)$$

In the first, second, and last of these series the absolute values of the terms are *decreasing;* that is, the absolute value of each term is less than the absolute value of the preceding term. Thus,

in (α): $$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \frac{1}{5} > \frac{1}{6} > \cdots,$$

in (β): $$1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{4}} > \frac{1}{\sqrt{5}} > \frac{1}{\sqrt{6}} > \cdots,$$

and in (ε): $$2 > 1\frac{1}{2} > 1\frac{1}{4} > 1\frac{1}{8} > 1\frac{1}{16} > \cdots.$$

26. LEIBNIZ' TEST

This may be stated as follows:

LEIBNIZ' TEST. *If the terms of an alternating series are decreasing in absolute value and approaching zero as a limit, then the series converges.*

On the basis of this test, the series (α) and (β) are convergent. In the series (ε), although the terms are decreasing, they do not approach zero (their absolute values clearly approach one). And since in every convergent series (and in particular in an alternating series), the terms must approach zero, as we have seen on pages 56–57, the series (ε) cannot converge.

Regarding the series (γ) and (δ), we see that in each of them the terms tend to zero, but the absolute values of the terms of these series do not decrease *monotonically.* In the series (γ), the absolute value of the 6th term is $\frac{1}{2^3} = \frac{1}{8}$, and the absolute value of the 7th is

$$\frac{1}{7} > \frac{1}{8}.$$

The next term has absolute value $\frac{1}{2^4} = \frac{1}{16}$, and the next

$$\frac{1}{9} > \frac{1}{16}.$$

In the series (δ), the absolute value of the first term is less than one, of the second less than $\frac{1}{2^2}$, of the third less than $\frac{1}{3^2}$, etc. In general, the absolute value of the nth term, equal to $\frac{\sin^2 n^\circ}{n^2}$, is no greater than $\frac{1}{n^2}$, since $\sin^2 n^\circ$ is never greater than one. But the absolute value of the 180th term is $\frac{\sin^2 180^\circ}{180^2} = 0$, and the absolute value of the 181st term is

$$\frac{\sin^2 181^\circ}{181^2} > 0.$$

Also, the absolute value of the 361st term is greater than the absolute value of the 360th term, and so forth. Therefore, Leibniz' test is not applicable to series (γ) and (δ), and without a special investigation we cannot say whether these series are convergent or not.

Let us now prove that Leibniz' test is valid.

Proof. We assume that we have an alternating series

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + u_7 - \cdots \qquad (a)$$

(here, u_1, u_2, u_3, u_4, ... represent positive numbers) in which the absolute values of the terms are decreasing, that is,

$$u_1 > u_2 > u_3 > u_4 > u_5 > u_6 > u_7 > \cdots$$

and approach zero, that is, $\lim_{n\to\infty} u_n = 0$. We must prove that the series (*a*) converges. The proof will be clear if we use the geometric representation of numbers. We shall lay off on a number line (Fig. 8) the

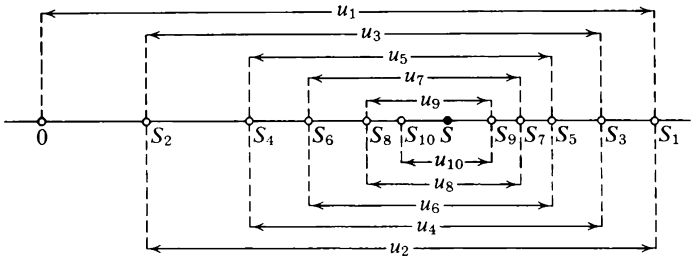


Fig. 8

points S_1, S_2, S_3, S_4, S_5, S_6, S_7, ..., representing the partial sums of our series. We obtain S_1 by laying off to the right of zero a segment equal to $u_1 (S_1 = u_1)$. The second partial sum $S_2 = u_1 - u_2$. Therefore, in order to obtain the point S_2, we lay off the segment u_2 to the left of the point S_1. Since we assumed $u_1 > u_2$, the point S_2 will be between 0 and S_1. The third partial sum is $S_3 = u_1 - u_2 + u_3 = S_2 + u_3$. Therefore the point S_3 is obtained by laying off to the right of S_2 a segment of length u_3. The point S_3 should be found between S_1 and S_2, since by assumption u_3 is smaller than u_2. In the same way, if we lay off to the left of the point S_3 the segment u_4, we obtain the point S_4, situated between S_2 and S_3, then the point S_5 between S_3 and S_4, etc. It is easy to see that in this way we obtain the picture shown in Figure 8; that is, the points representing the odd-numbered partial sums S_1, S_3, S_5, S_7, S_9, ... will be arranged one after another from right to left; the points S_2, S_4, S_6, S_8, S_{10}, ... representing the even-numbered partial sums will be arranged from left to right. In addition, any even-numbered point will lie to the

left of any odd-numbered point and finally, the distances between the points in the pairs S_1 and S_2, S_2 and S_3, S_3 and S_4, S_4 and S_5, . . . , S_{2k-1} and S_{2k}, S_{2k} and S_{2k+1}, . . . become smaller and smaller, approaching zero as a limit. This is because the absolute values of the above-mentioned distances are equal to the terms of the series (*a*), u_1, u_2, u_3, u_4, u_5, u_6, . . . , u_{2k}, u_{2k+1}, . . . , which by assumption approach zero.

It follows clearly from what we have said that there should exist a point S that would separate the odd-numbered points from the even-numbered points, so that the former lie to the right of S and the latter to the left of S (Fig. 8). It is also clear that there can be only one such point. For if there were two points S and S' which both separated the even- from the odd-numbered points, then the segment SS' would lie inside *every* segment S_1S_2, S_2S_3, S_3S_4, S_4S_5, S_5S_6, . . . , $S_{2k-1}S_{2k}$, $S_{2k}S_{2k+1}$, . . . , etc. But this is impossible, since, as we have seen, the lengths of these segments tend to zero (Fig. 9).

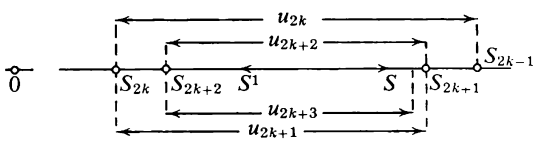


Fig. 9

Finally, it is clear that the partial sums S_1, S_2, S_3, S_4, S_5, S_6, . . . , S_{2k}, S_{2k+1}, . . . will approach the number S (corresponding to the point S) as a limit. In other words, the distances between the points representing the partial sums and the point S will tend to zero. This follows from the fact that the distance between the point S and either of the two points S_{2k} and S_{2k+1} is smaller than the distance between the points S_{2k} and S_{2k+1} (Fig. 10). But this distance equals u_{2k+1}, a

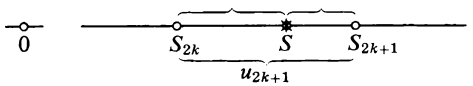


Fig. 10

quantity which by assumption tends to zero. Thus, we have shown that there exists a number S which the partial sums of the series (*a*) approach as a limit.

27. ESTIMATES OF ERROR

In the preceding proof we have seen that all even-numbered partial sums of the series (sums of an even number of terms) are smaller than the sum S and the odd-numbered sums are greater than S.

We can evaluate the error in approximating the sum of the series (*a*) by the partial sums S_{2k} and S_{2k+1}. In Figure 11 we see that

$$0 < S - S_{2k} < u_{2k+1}$$

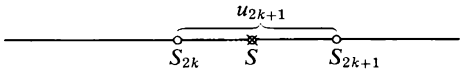


Fig. 11

and in Figure 12 that

$$0 < S_{2k+1} - S < u_{2k+2}.$$

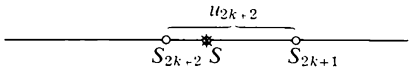


Fig. 12

We can summarize these two results as follows.

THEOREM. *If in an infinite alternating series whose terms are decreasing in absolute value we discard all terms following some given term, then the absolute value of the error we make in so approximating the sum of the series is smaller than the first rejected term. If the series starts with a positive term, the error is negative if the last term considered is even-numbered; it is positive if the last term considered is odd-numbered.*

EXAMPLE 1. Consider the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots.$$

We can approximate the sum of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots$$

from below by taking the first ten terms (since ten is an even number) with an error smaller than $\frac{1}{11}$ (since the first rejected term is $\frac{1}{11}$). Thus, in this example

$$S_{10} < S < S_{10} + \frac{1}{11} \approx S_{10} + 0.091.$$

Computing S_{10} we obtain

$$\begin{array}{rr} +\ 1 = 1.000 & -\frac{1}{2} = -\ 0.500 \\ +\frac{1}{3} = 0.333 & -\frac{1}{4} = -\ 0.250 \\ +\frac{1}{5} = 0.200 & -\frac{1}{6} = -\ 0.167 \\ +\frac{1}{7} = 0.143 & -\frac{1}{8} = -\ 0.125 \\ +\frac{1}{9} = 0.111 & -\frac{1}{10} = -\ 0.100 \\ \hline +\ 1.787 & -\ 1.142 \end{array}$$

$$S_{10} = 1.787 - 1.142 = 0.645.$$

Therefore,

$$0.645 < S < 0.645 + 0.091 = 0.736.$$

In the last chapter of this book, we shall show how to compute the sum of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots$$

with great accuracy. We shall see that it is equal to 0.69314718 . . . .

EXAMPLE 2. We shall now consider the following alternating series:

$$1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \frac{1}{6^6} + \frac{1}{7^7} - \frac{1}{8^8} + \cdots .$$

Evidently the absolute values of its terms decrease ($1 > \frac{1}{2^2} > \frac{1}{3^3} > \frac{1}{4^4} > \frac{1}{5^5} > \frac{1}{6^6} > \cdots$) and approach zero. By Leibniz' test this series converges. We shall approximate its sum with an error less than 0.00005. For such accuracy, we must select as the first discarded term one whose absolute value is less than $0.00005 = \frac{1}{20000}$; that is, a term represented by a fraction with a denominator greater than 20,000. But $2^2 = 4$, $3^3 = 27$, $4^4 = 256$, $5^5 = 3125$ and $6^6 = 46656$. Thus the sixth term of our series, $-\frac{1}{6^6}$, satisfies our demand. So, using the sum of the first five terms of the series (which approximates the sum S of the series from above, since five is odd), we can assert that

$$S_5 > S > S_5 - \frac{1}{46656} \approx S_5 - 0.000021$$

(where $-\frac{1}{46656} = -\frac{1}{6^6}$ is the first discarded term of the series).

Computing S_5 we obtain

$$\begin{array}{rr} 1 = & 1.000000 \\ \frac{1}{3^3} = \frac{1}{27} = & 0.037037 \\ \frac{1}{5^5} = \frac{1}{3125} = & 0.000320 \\ \hline & 1.037357 \end{array} \qquad \begin{array}{rr} -\frac{1}{2^2} = -\frac{1}{4} = & -0.250000 \\ -\frac{1}{4^4} = -\frac{1}{256} = & -0.003906 \\ \hline & -0.253906 \end{array}$$

$$S_5 = 1.037357 - 0.253906 = 0.783451.$$

Hence,

$$0.783451 > S > 0.783430.$$

Therefore we can take $S = 0.783440$ with an error not greater than 0.000011.

28. SERIES OF CONSTANT SIGN

We shall now consider infinite series of constant sign, that is, series all of whose terms have the same sign. Clearly, here we need only consider series with positive terms since, in case of a series with all terms negative, it is possible to factor out the minus sign in front of each partial sum and to deal with the series as if it had all terms positive.

We have seen (p. 58) that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

is an example of a series with positive terms approaching zero although the series does not converge. For this reason the study of series with positive terms is more complicated than the study of alternating series with decreasing terms, as there the problem was completely solved if the limit of the terms existed and was equal to zero.

29. D'ALEMBERT'S TEST

For series with positive terms we shall obtain a convenient (but not always applicable) test, if, instead of considering the behavior of only one term of the series as its index increases, we consider the behavior of the ratio of a pair of consecutive terms. This is suggested by the well-known example of a geometric series

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_n + u_{n+1} + \cdots,$$

where

$$u_1 = a, \quad u_2 = aq, \quad u_3 = aq^2, \quad u_4 = aq^3, \quad u_5 = aq^4, \ldots,$$
$$u_n = aq^{n-1}, \quad u_{n+1} = aq^n, \ldots.$$

Here the ratio of two consecutive terms—the following to the preceding—remains the same for all terms of the series

$$\frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \frac{u_5}{u_4} = \cdots = \frac{u_{n+1}}{u_n} = \cdots = q.$$

And we know that for $0 < q < 1$ the infinite series (geometric progression) converges and its sum is

$$S = \frac{a}{1 - q},$$

while for $q \geq 1$ the series diverges.

It is possible to establish something similar for other series with positive terms. In general, in a series

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_n + u_{n+1} + \cdots$$

the ratio of a following term to the preceding one will not remain constant but will change from term to term.

EXAMPLE 1. In the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots,$$

we have

$$\frac{u_2}{u_1} = \frac{\frac{1}{2}}{1} = \frac{1}{2}, \qquad \frac{u_3}{u_2} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}, \qquad \frac{u_4}{u_3} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4},$$

$$\frac{u_5}{u_4} = \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{4}{5}, \quad \cdots, \quad \frac{u_{n+1}}{u_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1},$$

EXAMPLE 2. In the series

$$1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots$$
$$+ \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot \ldots \cdot n} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5 \cdots (n+1)} + \cdots,$$

these ratios are

$$\frac{u_2}{u_1} = \frac{\frac{1}{1\cdot 2}}{1} = \frac{1}{2}, \qquad \frac{u_3}{u_2} = \frac{\frac{1}{1\cdot 2\cdot 3}}{\frac{1}{1\cdot 2}} = \frac{1}{3},$$

$$\frac{u_4}{u_3} = \frac{\frac{1}{1\cdot 2\cdot 3\cdot 4}}{\frac{1}{1\cdot 2\cdot 3}} = \frac{1}{4}, \qquad \frac{u_5}{u_4} = \frac{\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}}{\frac{1}{1\cdot 2\cdot 3\cdot 4}} = \frac{1}{5}, \qquad \cdots,$$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{1}{1\cdot 2\cdot 3\cdot \ldots \cdot n(n+1)}}{\frac{1}{1\cdot 2\cdot 3\cdot \ldots \cdot n}} = \frac{1}{n+1}.$$

D'ALEMBERT'S RATIO TEST. *A series of positive terms converges if the ratio of a term to its predecessor is never greater than some number q which is less than one, that is, if there exists a q such that*

$$\frac{u_{n+1}}{u_n} \leq q < 1$$

for all n. The series diverges if this ratio is not less than one, that is, if

$$\frac{u_{n+1}}{u_n} \geq 1.^1$$

D'ALEMBERT (1717–1783)

We should remark that in applying this test it is not enough to know that the ratio $\frac{u_{n+1}}{u_n}$ is always less than one. The test requires that this ratio be less than some constant number q less than one. In Example 1 (the harmonic series) the ratios, taken in order, are

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \cdots, \frac{n}{n+1},$$

They are all less than one but they approach one as a limit. Therefore there exists no number q less than one which could separate these

[1] In this general form, the theorem was formulated and proved by Cauchy in his book *Algebraic Analysis* (1821). D'Alembert himself in his *Mathematical Remarks* (*Opuscules mathématiques,* 1768) investigated ratios of a term to the preceding one only for the binomial series. If the absolute value of this ratio is less than one, that is, the next term is less than the preceding one, then he called the series "convergent"; if greater than one he called it "divergent." From this point of view we would call the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$ "convergent" although it has no sum (p. 58).

A series may be "convergent" at the beginning and "divergent" at the end and vice versa. D'Alembert gives an example of the binomial expansion $(1 + x)^{\frac{1}{2}}$ for $x = \frac{200}{199}$. He shows that the first 300 terms of this expansion are decreasing (that is, that we may consider them "convergent") and that, starting from term 302, they increase (that is, according to d'Alembert, "the series begins to diverge"). In the second example, the expansion of $(1 + x)^{-2}$ for $x = \frac{99}{100}$, the corresponding series "diverges up to the 99th term" and then becomes "convergent."

From this, it is evident that d'Alembert was far from giving a criterion of convergence of the infinite series in the sense that Cauchy understood it and as we understand it today. The test is called d'Alembert's test, though, because in it we consider the ratio of a term of a series to the preceding one, and d'Alembert deserves the credit for the close investigation of this ratio (even if only in a particular case).

ratios from one. Indeed, if such a number $q (q < 1)$ existed, greater than all the ratios, then they all would differ from one by more than $1 - q$ and hence could not approach one as a limit. Thus, d'Alembert's test does not permit us to draw any conclusion about the harmonic series. We proved earlier (on p. 58) that the harmonic series is divergent.

In Example 2 (the series $1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$) the ratios of a term to the preceding one, taken in order, are

$$\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \frac{1}{7}, \quad \cdots, \quad \frac{1}{n+1},$$

We see that each of them is no greater than $\frac{1}{2}$. Thus,

$$\frac{u_{n+1}}{u_n} \leq \frac{1}{2} < 1,$$

and by d'Alembert's test the series converges.

EXAMPLE 3. Let us consider the infinite series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots.$$

Here the ratios of following terms to the preceding, taken in order, are

$$\frac{\frac{1}{2^2}}{\frac{1}{1}} = \frac{1}{4}, \quad \frac{\frac{1}{3^2}}{\frac{1}{2^2}} = \frac{4}{9}, \quad \frac{\frac{1}{4^2}}{\frac{1}{3^2}} = \frac{9}{16},$$

$$\frac{\frac{1}{5^2}}{\frac{1}{4^2}} = \frac{16}{25}, \quad \frac{\frac{1}{6^2}}{\frac{1}{5^2}} = \frac{25}{36}, \quad \cdots,$$

$$\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{n^2}{n^2 + 2n + 1},$$

It is clear at once that all these ratios are less than one. It is less clear that they approach one as a limit. This will become clear, however, if we write the ratio of the $(n + 1)$st term to the nth, equal to

$$\frac{n^2}{n^2 + 2n + 1},$$

in the form

$$\frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}}.$$

(We divided the numerator and the denominator of the fraction by n^2.) If n increases without bound, then $\frac{2}{n}$ and $\frac{1}{n^2}$ approach zero,

$$1 + \frac{2}{n} + \frac{1}{n^2}$$

approaches one and, consequently, the fraction

$$\frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}}$$

approaches one. Thus there can here be no number q less than one and greater than all the ratios. D'Alembert's test is therefore inapplicable to this series. On pages 60–61, using Cauchy's criterion, we showed that this series converges.

Comparing the results of Example 1 and Example 3, we see that when

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1,$$

the series may be either convergent (Example 3) or divergent (Example 1).

Now we shall turn to the proof of d'Alembert's test.

Proof (*part 1*). We shall assume the inequalities

$$\frac{u_2}{u_1} \leq q, \quad \frac{u_3}{u_2} \leq q, \quad \frac{u_4}{u_3} \leq q, \quad \frac{u_5}{u_4} \leq q \quad \cdots, \quad \frac{u_{n+1}}{u_n} \leq q,$$

($q < 1$). From these inequalities it follows that

$$u_2 \leq qu_1, \quad u_3 \leq qu_2, \quad u_4 \leq qu_3, \quad u_5 \leq qu_4, \quad \cdots, \quad u_{n+1} \leq qu_n, \quad \cdots.$$

Since $u_3 \leq qu_2$ and $u_2 \leq qu_1$, then $u_3 \leq q$ (qu_1), i.e. $u_3 \leq q^2u_1$. Furthermore, $u_4 \leq qu_3$; using the inequality $u_3 \leq q^2u_1$ we obtain $u_4 \leq q(q^2u_1)$ or $u_4 \leq q^3u_1$. In the same way, we obtain $u_5 \leq q^4u_1$, $u_6 \leq q^5u_1$, etc. In general,

$$u_{n+1} \leq q^nu_1, \tag{1}$$

for any number n. (Let us remark that if

$$\frac{u_2}{u_1} \leq q, \quad \frac{u_3}{u_2} \leq q, \quad \frac{u_4}{u_3} \leq q, \quad \cdots, \quad \frac{u_{n+1}}{u_n} \leq q,$$

could be replaced by

$$\frac{u_2}{u_1} = q, \quad \frac{u_3}{u_2} = q, \quad \frac{u_4}{u_3} = q, \quad \cdots, \quad \frac{u_{n+1}}{u_n} = q, \quad \cdots,$$

that is, if the given series were a geometric series, then the inequalities proved above,

$$u_2 \leq qu_1, \quad u_3 \leq q^2u_1, \quad u_4 \leq q^3u_1, \quad \cdots, \quad u_{n+1} \leq q^nu_1, \quad \cdots,$$

would become the well-known formulas for the terms of a geometric series

$$u_2 = qu_1, \quad u_3 = q^2u_1, \quad u_4 = q^3u_1, \quad \cdots, \quad u_{n+1} = q^nu_1, \quad \cdots.)$$

Having established this, we can now use Cauchy's criterion. For this purpose, we shall consider the sum of an arbitrary number p of terms starting with the $(n + 1)$st

$$u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p}.$$

Since

$$u_{n+1} \leq q^nu_1, \quad u_{n+2} \leq q^{n+1}u_1, \quad u_{n+3} \leq q^{n+2}u_1, \quad \cdots, \quad u_{n+p} \leq q^{n+p-1}u_1,$$

this sum is less than or equal to

$$q^n u_1 + q^{n+1}u_1 + q^{n+2}u_1 + \cdots + q^{n+p-1}u_1$$
$$= q^n u_1(1 + q + q^2 + \cdots + q^{p-1})$$
$$= q^n u_1 \frac{1 - q^p}{1 - q} < q^n u_1 \frac{1}{1 - q}.$$

We find that

$$u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p} < \frac{q^n u_1}{1 - q}.$$

But the right-hand side is another form of the product

$$\frac{u_1}{1 - q} \cdot q^n.$$

The first factor does not change as n increases. But the second factor, q^n, as the nth power of a positive number less than one, becomes smaller as n increases, and approaches zero as n increases indefinitely. So the right side, and *a fortiori* the left side of the inequality approaches zero. Thus, we have shown that the sum

$$u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p}$$

can be made as small as we wish for all sufficiently large n and all positive p. And this means, by Cauchy's criterion, that the series

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_n + u_{n+1} + \cdots$$

converges, so the first part is proved.

Proof (part 2). We still have to prove that for $\frac{u_{n+1}}{u_n} \geq 1$ the infinite series diverges. This proof is easy. Since

$$\frac{u_2}{u_1} \geq 1, \quad \frac{u_3}{u_2} \geq 1, \quad \frac{u_4}{u_3} \geq 1, \quad \frac{u_5}{u_4} \geq 1, \quad \cdots, \quad \frac{u_{n+1}}{u_n} \geq 1, \quad \cdots,$$

we have

$$u_2 \geq u_1, \quad u_3 \geq u_2, \quad u_4 \geq u_3, \quad u_5 \geq u_4, \quad \cdots, \quad u_{n+1} \geq u_n, \quad \cdots,$$

or

$$u_1 \leq u_2 \leq u_3 \leq u_4 \leq u_5 \leq \cdots \leq u_n \leq u_{n+1} \cdots.$$

This means that all the terms of the series are greater than or equal to the positive number u_1 and therefore cannot approach zero. But in a convergent series the terms must approach zero. Consequently this series does not converge, and the proof is complete.

In applying d'Alembert's test the inequality

$$\frac{u_{n+1}}{u_n} \leq q < 1$$

or

$$\frac{u_{n+1}}{u_n} \geq 1$$

need not hold for all ratios starting from the first one. *It is sufficient for one of the above inequalities to hold for all terms starting from any one* (from the third, the tenth, the thousandth, etc.).

This assertion is correct because the first terms of a series are not relevant to its convergence or divergence. In other words, we can prefix a given series with as many new terms as we wish or discard an arbitrary number of the first terms of a series (the first 10, 500, 1000 terms of a series) and still the new series will converge if the original series converged (the sum will be changed by the sum of the terms added or discarded) and will diverge if the original series diverged.

EXAMPLE 4. In the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots,$$

the ratios of the terms to their predecessors are

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6},$$

and we cannot say that they are *all* less than some number q less than one (since the number one is among them). But starting with the third term, all the terms are smaller than $\frac{1}{2}$. Therefore, d'Alembert's test is applicable and we may assert that the series converges. (Clearly, this series is obtained from the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots,$$

which we investigated in detail in Example 2, but with the addition of the single term, 1.)

EXAMPLE 5. In the series

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{2}{25} + \frac{3}{26} + \frac{4}{27} + \frac{5}{28} + \frac{6}{29} + \frac{7}{30} + \frac{8}{31} + \cdots$$

the ratios of the terms are

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{48}{25}, \frac{75}{52}, \frac{104}{81}, \frac{135}{112}, \frac{168}{145}, \frac{203}{180}, \frac{240}{217},$$

It is incorrect to say that they are all greater than one. But the fourth ratio and all following ones are greater than one. From this it follows that the series diverges.

30. ESTIMATES OF ERROR

The proof we have given for part 1 of d'Alembert's test not only shows that the series is convergent (that is, that there exists a definite number S which the partial sums $S_1, S_2, S_3, S_4, \ldots, S_n, \ldots$ approach as a limit), but also provides a method for estimating the error we make in taking the partial sum

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

in place of the sum S of the series. This error is equal to the sum of the discarded terms, i.e., to the sum of the series

$$u_{n+1} + u_{n+2} + u_{n+3} + u_{n+4} + \cdots .$$

And since for an arbitrary p

$$u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p} < \frac{q^n u_1}{1 - q},$$

it follows that the sum of the series $u_{n+1} + u_{n+2} + u_{n+3} + u_{n+4} + \cdots$ will be no greater than $\frac{q^n u_1}{1 - q}$; that is,

$$u_{n+1} + u_{n+2} + u_{n+3} + u_{n+4} + \cdots \leq \frac{q^n u_1}{1 - q}. \tag{2}$$

Thus the error caused by discarding all the terms starting with the $(n + 1)$st is not greater than

$$\frac{q^n u_1}{1 - q}.$$

We return to the series of Example 2 in section 29,

$$1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots .$$

We have already seen (p. 70) that here the ratios of each term to the preceding one are no greater than $\frac{1}{2}$ and, consequently, this series is convergent. Furthermore, on the basis of inequality (2) we may assert that the error incurred in computing the sum of the series from its first eight terms

$$1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}$$

is less than $\dfrac{q^8 u_1}{1-q} = \dfrac{\left(\frac{1}{2}\right)^8 \cdot 1}{1-\frac{1}{2}} = \dfrac{\left(\frac{1}{2}\right)^8}{\frac{1}{2}} = \left(\frac{1}{2}\right)^7 = \frac{1}{128} \approx 0.008.$

In fact, the error is equal to the sum of the series

$$\frac{1}{9!} + \frac{1}{10!} + \frac{1}{11!} + \frac{1}{12!} + \frac{1}{13!} + \cdots \qquad (3)$$

This sum will be a number considerably less than 0.008. Indeed, in the series (3), formed from the discarded terms, the ratio of each term to its predecessor is

$$\frac{\frac{1}{10!}}{\frac{1}{9!}} = \frac{1}{10}, \quad \frac{\frac{1}{11!}}{\frac{1}{10!}} = \frac{1}{11}, \quad \frac{\frac{1}{12!}}{\frac{1}{11!}} = \frac{1}{12},$$

None of these is greater than $\frac{1}{10}$. Therefore, for the series (3), we can set $q = \frac{1}{10}$. Evidently, for the terms of the series (3), we shall have

$$\frac{1}{10!} = \frac{1}{10}\cdot\frac{1}{9!}, \quad \frac{1}{11!} < \left(\frac{1}{10}\right)^2 \cdot \frac{1}{9!}, \quad \frac{1}{12!} < \left(\frac{1}{10}\right)^3 \cdot \frac{1}{9!},$$

(Compare these with the inequalities (1) established in the general case p. 74.) But it follows from this that

$$\frac{1}{9!}+\frac{1}{10!}+\frac{1}{11!}+\frac{1}{12!}+\cdots$$

$$<\frac{1}{9!}+\frac{1}{10}\cdot\frac{1}{9!}+\left(\frac{1}{10}\right)^2\cdot\frac{1}{9}+\left(\frac{1}{10}\right)^3\cdot\frac{1}{9!}+\cdots$$

$$=\frac{1}{9!}\left[1+\frac{1}{10}+\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^3+\cdots\right]=\frac{1}{9!}\cdot\frac{1}{1-\frac{1}{10}}$$

$$=\frac{1}{9!}\cdot\frac{10}{9}=\frac{10}{362{,}880\cdot 9}=\frac{1}{326{,}592}\approx 0.000003.$$

Therefore, approximating the sum of the series

$$1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+\cdots$$

by the sum of its first eight terms only, we make an error not greater than 0.000003. To avoid making this small error worse by the errors in changing the terms of the series into decimals, we should take the latter to the eighth digit after the decimal point. We obtain

$$\begin{aligned}
1 &= 1.00000000\\
\frac{1}{2!} &= 0.50000000\\
\frac{1}{3!} &= 0.16666667\\
\frac{1}{4!} &= 0.04166667\\
\frac{1}{5!} &= 0.00833333\\
\frac{1}{6!} &= 0.00138889\\
\frac{1}{7!} &= 0.00019841\\
\frac{1}{8!} &= 0.00002480\\
\hline
S_8 &= 1.71827877.
\end{aligned}$$

(The computations here are made easier because the number in the second row is obtained from the number in the first row by dividing the latter by 2; the number in the third row by dividing the number in the second row by 3; and moving from each remaining row to the next by dividing the numbers by 4, 5, 6, 7, and 8.)

Thus, the sum of the series is $S = 1.718279$ to within 0.000003 and we see that the sum of the series of Example 4 in section 29 is 2.718279.

31. ABSOLUTELY AND CONDITIONALLY CONVERGENT SERIES

We still have to say something about the convergence of series whose terms have arbitrary signs (that is, not necessarily alternating or all the same). It is often possible to reduce a question about the convergence of such general series to a question about the convergence of a series with positive terms. Instead of considering a given series

$$u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + \cdots, \qquad (*)$$

we consider the series of the absolute values of the terms of this series

$$|u_1| + |u_2| + |u_3| + |u_4| + |u_5| + |u_6| + \cdots. \qquad (**)$$

The terms of the second series are all positive. If by any method (for example, d'Alembert's test) we are able to show that the series (**) converges, *then we can assert that the series* (*) *also converges.* (Of course, the series (*) generally has a different sum from that of series (**).) To prove this, we note that the series (**) converges, so that by Cauchy's criterion the sum

$$|u_{n+1}| + |u_{n+2}| + |u_{n+3}| + \cdots + |u_{n+p}|$$

can be made less than any given positive number ε, however small, by making n sufficiently large (regardless of the value of p). But then the absolute value of the sum of the corresponding terms of the first series (*)

$$|u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p}|$$

will be less than ε, since

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| \leq |u_{n+1}| + |u_{n+2}| + |u_{n+3}| + \cdots + |u_{n+p}|.$$

(This is true because among the numbers $u_{n+1}, u_{n+2}, u_{n+3}, \ldots, u_{n+p}$ in the left-hand part of this inequality there may be numbers with different signs whose addition would then be equivalent to the *subtraction* of their absolute values. In the right-hand part of the inequality all the terms are positive and therefore their absolute values are all added together.) Thus we find that the sum

$$|u_{n+1} + u_{n+2} + u_{n+3} + \cdots + u_{n+p}|,$$

regardless of p, can be made arbitrarily small by making n sufficiently

large. But from this, again by Cauchy's criterion, there follows the convergence of the given series

$$u_1 + u_2 + u_3 + u_4 + \cdots + u_n + u_{n+1} + \cdots .$$

EXAMPLE 1. We shall determine whether the following series is convergent:

$$1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} - \frac{1}{9^2} - \frac{1}{10^2}$$
$$+ \frac{1}{11^2} + \frac{1}{12^2} + \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} - \frac{1}{16^2} - \frac{1}{17^2} - \frac{1}{18^2}$$
$$- \frac{1}{19^2} - \frac{1}{20^2} - \frac{1}{21^2} + \frac{1}{22^2} + \cdots .$$

The series of absolute values of the terms of the given series is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots .$$

This series converges as was shown in Example 3 of section 29. Consequently, the given series also converges.

EXAMPLE 2. Let us consider the series

$$1 - \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} - \frac{1}{10!} - \frac{1}{11!} + \cdots .$$

The series of absolute values of its terms

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \cdots$$

converges. Consequently, the given series also converges.

If for a series

$$u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + \cdots + u_n + u_{n+1} + \cdots \quad (*)$$

the series of absolute values of its terms

$$|u_1| + |u_2| + |u_3| + |u_4| + |u_5| + \cdots$$
$$+ |u_n| + |u_{n+1}| + \cdots \quad (**)$$

converges, then we call the given series *absolutely convergent.* For instance, the series of Example 1 and 2 are absolutely convergent.

Not every convergent series is absolutely convergent. Indeed, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

converges (in accordance with Leibniz's test) while the series of absolute values of its terms

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots$$

diverges. Thus, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

converges, but not absolutely. We say it converges *conditionally*.

The chief difference between the properties of absolutely convergent and conditionally convergent series is that *the sum of a conditionally convergent series may change as a result of a rearrangement of the terms, but the sum of an absolutely convergent series is unchanged by any rearrangement of its terms.* Leaving aside a detailed analysis of these facts, we shall present here an example of a change in the sum of a conditionally convergent series caused by a rearrangement of its terms.

EXAMPLE 3. Let us consider the conditionally convergent series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots . \qquad (a)$$

We saw on page 67 that its sum lies between 0.645 and 0.736. More accurate computation shows that this sum is 0.69314718 .... Let us rearrange the terms of the series (without changing any of them individually) so that one negative term follows two positive terms. We obtain the series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \frac{1}{17} + \frac{1}{19} - \frac{1}{10} + \cdots . \qquad (a')$$

We shall show that the sum of this series is different from the sum of the original series.

To show this, we observe that series (a') can be obtained from series (a) by adding to it the series

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots . \qquad (b)$$

We write these two series one under another, thus:

$$\begin{aligned} &1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \\ &+ \frac{1}{2} \qquad - \frac{1}{4} \qquad + \frac{1}{6} \qquad - \frac{1}{8} \\ &\qquad\qquad - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \cdots \\ &\qquad\qquad + \frac{1}{10} \qquad - \frac{1}{12} \qquad + \frac{1}{14} + \cdots \end{aligned}$$

and add the terms standing one under another, obtaining

$$\begin{aligned} &1 + 0 + \frac{1}{3} - \frac{2}{4} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{2}{8} + \frac{1}{9} \\ &\qquad\qquad + 0 + \frac{1}{11} - \frac{2}{12} + \frac{1}{13} + 0 + \cdots , \end{aligned}$$

that is,

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{3} + \frac{1}{15} - \frac{1}{8} + \cdots .$$

But the sum of series (a) is 0.69314718 . . . , and the sum of series (b), whose terms are the halves of the terms of series (a), is $\frac{1}{2}(0.69314718\ldots) = 0.34657359\ldots$. Therefore, the sum of the series formed by their addition, series (a'), which differs from (a) only in the order of its terms, is $0.69314718 + 0.34657359 = 1.03972077$ and is thus different from the sum of the original series, (a).

The conclusion in Example 3, paradoxical at first glance, teaches that infinite series, that is, sums of indefinitely increasing numbers of terms, have different properties from those of ordinary sums (of a finite number of terms). Mathematicians of the 18th century tended to ignore this situation and to treat series as finite sums. They consequently arrived at some false conclusions.

32. OPERATIONS WITH SERIES

In the first half of the 19th century Cauchy, Abel, Dirichlet, and Riemann established by thorough but simple investigations that only in absolutely convergent series can we arbitrarily change the order of terms and carry out all operations (addition, subtraction, multiplication, division, raising to a power) in the same way as if they were ordinary finite sums or polynomials.

Given a conditionally convergent series, we can rearrange the terms to make the sum of the new series equal to any arbitrary number and even make the new series diverge. Hence, we must be cautious in all operations with conditionally convergent series. For instance, multiplication of conditionally convergent series can, in general, be performed only in a strictly determined order, which frequently makes it difficult to group like terms in the outcome. We shall keep these remarks in mind and, before we carry out any operations on series, verify that the series not only converge but converge absolutely.

We shall now derive the rule for multiplication of absolutely convergent series.

Let

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots = a$$

and

$$v_1 + v_2 + v_3 + \cdots + v_n + \cdots = b$$

be two absolutely convergent series. We shall show that by multiplying them term by term and adding the products we obtain a new convergent series (called the product of the given series), whose sum equals $a \cdot b$. Before we start our proof let us explain the method of obtaining products of series. We multiply each term of the first series by v_1 and write the products in one line; in the next line we indent and write the products of the first series by v_2; in the third line we indent some more and write the products of the first series by v_3, and so forth. We obtain the table

$$\begin{aligned} u_1 \cdot v_1 + u_2 \cdot v_1 + u_3 \cdot v_1 + u_4 \cdot v_1 + \cdots \\ u_1 \cdot v_2 + u_2 \cdot v_2 + u_3 \cdot v_2 + \cdots \\ u_1 \cdot v_3 + u_2 \cdot v_3 + \cdots \\ u_1 \cdot v_4 + \cdots \end{aligned}$$

If in this table we add the entries in the columns, we obtain a new series,

$$u_1 \cdot v_1 + (u_2 \cdot v_1 + u_1 \cdot v_2) + (u_3 \cdot v_1 + u_2 \cdot v_2 + u_1 \cdot v_3) + (u_4 \cdot v_1 + u_3 \cdot v_2 + u_2 \cdot v_3 + u_1 \cdot v_4) + \cdots, \qquad (***)$$

which is called the product of the given series. However, it is not at all clear that this series converges and that its sum is equal to the product $a \cdot b$ of the sums of the given series. The proof requires a rather long discussion.

First we shall consider the series consisting of the absolute values of the terms of the given series:

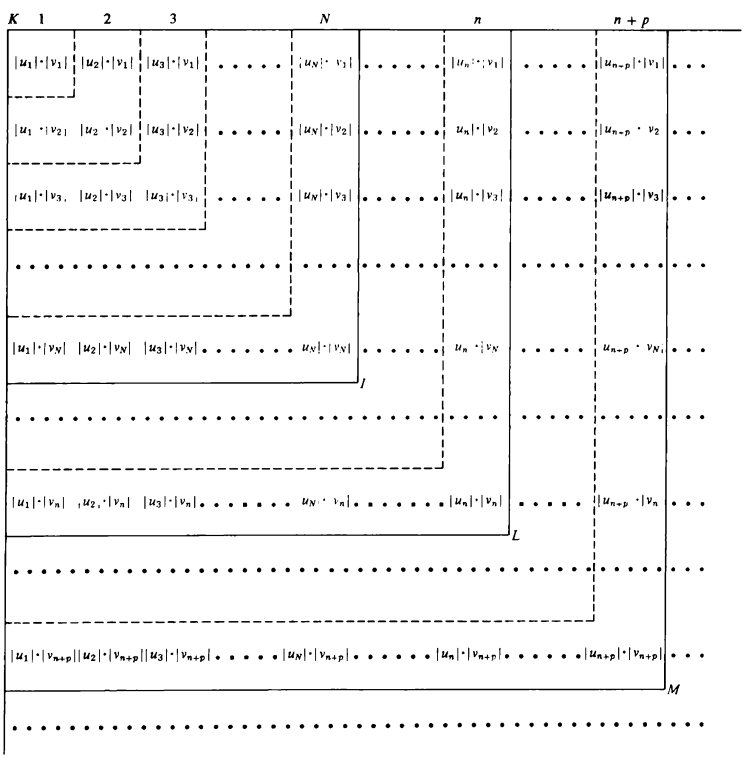
$$|u_1| + |u_2| + |u_3| + \cdots + |u_n| + \cdots = A$$

and

$$|v_1| + |v_2| + |v_3| + \cdots + |v_n| + \cdots = B.$$

(They converge because the given series are absolutely convergent.) We form the products of pairs of terms of these two series and arrange them as shown in Table I:

TABLE I



In this table we distinguish the square part KL, which consists of the intersection of the first n rows and first n columns where n is any given positive integer. Let us compute the sum of these terms and denote it by S_n. It is clear that the sum of the products lying in the first row of the distinguished part is

$$(|u_1| + |u_2| + |u_3| + \cdots + |u_n|) \cdot |v_1|,$$

in the second

$$(|u_1| + |u_2| + |u_3| + \cdots + |u_n|) \cdot |v_2|,$$

in the third

$$(|u_1| + |u_2| + |u_3| + \cdots + |u_n|) \cdot |v_3|,$$

etc., in the nth row

$$(|u_1| + |u_2| + |u_3| + \cdots + |u_n|) \cdot |v_n|.$$

Therefore, S_n is equal to the product of the two sums

$$S_n = (|u_1| + |u_2| + |u_3| + \cdots + |u_n|) \cdot (|v_1| + |v_2| + \cdots + |v_n|).$$

When n increases indefinitely, the first factor approaches the limit A, the second approaches the limit B, and, consequently, the whole product approaches the limit $A \cdot B$. But S_n can be considered as a partial sum of n terms of a series whose individual terms we obtain by adding products contained in the elbow-shaped parts of our table headed by the numbers $1, 2, 3, \ldots, n, \ldots$:

$$S_n = |u_1| \cdot |v_1| + (|u_2| \cdot |v_1| + |u_2| \cdot |v_2| + |u_1| \cdot |v_2|) + (|u_3| \cdot |v_1| + |u_3| \cdot |v_2| + |u_3| \cdot |v_3| + |u_2| \cdot |v_3| + |u_1| \cdot |v_3|) + \cdots.$$

Consequently, this series converges and its sum is $A \cdot B$. By Cauchy's criterion (p. 56) the difference between two partial sums $S_{n+p} - S_n$ can be made arbitrarily small for all sufficiently large n and any positive p. Thus, given any positive number ε, we can find a natural number N, such that

$$S_{n+p} - S_n < \frac{\varepsilon}{2}$$

for all $n \geq N$.

Since S_n is the sum of the products contained in the square KL and S_{n+p} is the sum of the products contained in KM, then $S_{n+p} - S_n$ is the sum of the products contained in the elbow-shaped figure LM.

In view of what we have said, this sum will be less than $\frac{\varepsilon}{2}$ if $n > N$ and p is any number.

From this it also follows that the sum of any number of products from Table I situated outside the square KI will also be less than $\frac{\varepsilon}{2}$. Given a finite set of such products, it is always possible to find a number $n > N$ and a large number p such that all these products will lie in the corresponding elbow-shaped figure LM. As each product is a nonnegative number, their sum will not be greater than the sum of all the products which fill up the elbow-shaped figure, that is, will be less than $\frac{\varepsilon}{2}$.

Let us return to the original series

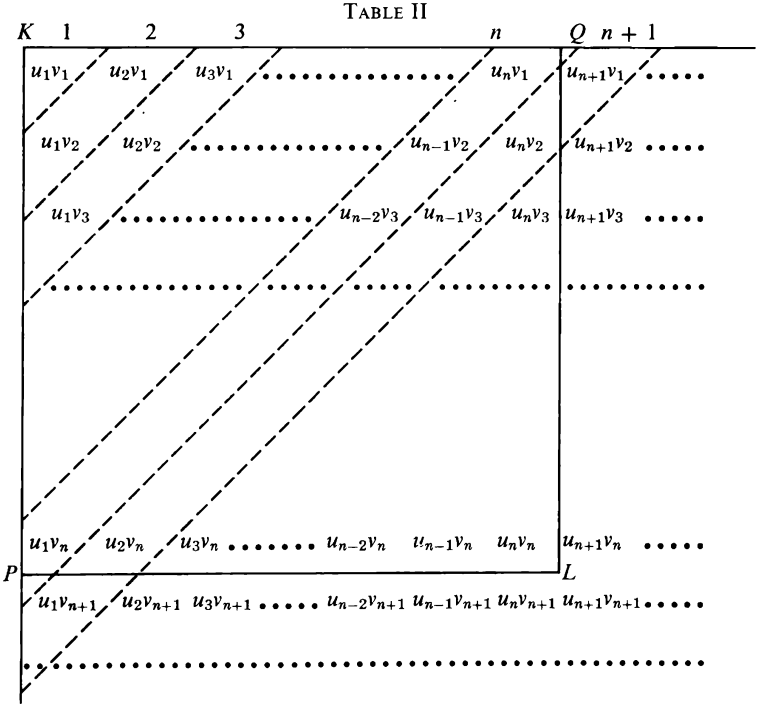
$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots = a$$

and

$$v_1 + v_2 + v_3 + \cdots + v_n + \cdots = b$$

and draw up Table II containing all the products of pairs of terms of these series:

TABLE II


K 1 2 3 n Q n + 1
u_1v_1 u_2v_1 u_3v_1 u_nv_1 $u_{n+1}v_1$
u_1v_2 u_2v_2 $u_{n-1}v_2$ u_nv_2 $u_{n+1}v_2$
u_1v_3 $u_{n-2}v_3$ $u_{n-1}v_3$ u_nv_3 $u_{n+1}v_3$
u_1v_n u_2v_n u_3v_n $u_{n-2}v_n$ $u_{n-1}v_n$ u_nv_n $u_{n+1}v_n$
P L
u_1v_{n+1} u_2v_{n+1} u_3v_{n+1} $u_{n-2}v_{n+1}$ $u_{n-1}v_{n+1}$ u_nv_{n+1} $u_{n+1}v_{n+1}$

Clearly Table I can be obtained from Table II by replacing every product in the latter by its absolute value. Therefore, the absolute value of the sum of an arbitrary set of products in Table II does not exceed the sum of the corresponding products in Table I.

The sum σ_n of the products in Table II contained in the triangle KPQ can be written in the form

$$\sigma_n = u_1 \cdot v_1 + (u_2 \cdot v_1 + u_1 \cdot v_2) + (u_3 \cdot v_1 + u_2 \cdot v_2 + u_1 \cdot v_3) + \cdots + (u_n v_1 + u_{n-1} v_2 + \cdots + u_1 v_n).$$

It is obvious that this sum coincides with the partial sum of n terms of the series (***). To show that $\sigma_n \to a \cdot b$ as $n \to \infty$, we observe first that σ_n differs from s_n, the sum of all the products of Table II contained in the square KL, by the sum of the products contained in the triangle PLQ. But the absolute value of the last sum is not greater than the sum of the corresponding products of Table I. If we take $n > 2N$ then, as we can see from Figure 13, all such products will be

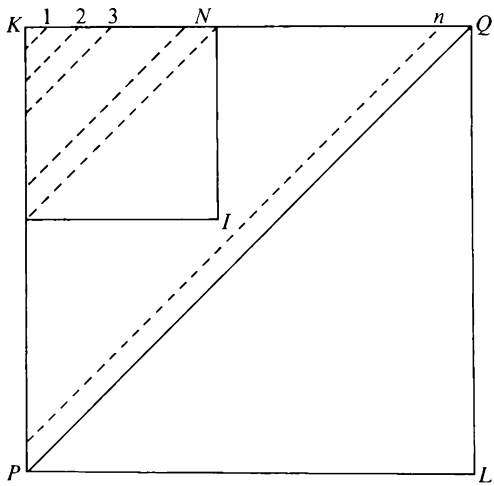


Fig. 13

outside the square KI. Therefore, the sum of the corresponding products in Table I will be less than $\frac{\varepsilon}{2}$ and, consequently, the absolute value of the sum of the products contained in triangle PLQ in Table II

will also be less than $\frac{\varepsilon}{2}$. Thus

$$|\sigma_n - S_n| < \frac{\varepsilon}{2}.$$

We can determine the sum s_n of all the products contained in the square KL in Table II in the same way as we computed S_n. Namely, we compute the sum of the products in the first row, then in the second and so on to the nth row, and then we add these sums. We obtain

$$s_n = (u_1 + u_2 + u_3 + \cdots + u_n)(v_1 + v_2 + v_3 + \cdots + v_n).$$

When n increases indefinitely, the first of these factors approaches the limit a, the second factor approaches the limit b, and s_n approaches $a \cdot b$. Therefore, for all sufficiently large n (say all $n \geq N_1$), we shall have

$$|s_n - a \cdot b| < \frac{\varepsilon}{2}.$$

Thus for a sufficiently large n ($n > 2N$ and $n \geq N_1$) σ_n differs from s_n by less than $\frac{\varepsilon}{2}$, and s_n differs from $a \cdot b$ also by less than $\frac{\varepsilon}{2}$. But, from this it follows that σ_n differs from $a \cdot b$ by less than $\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

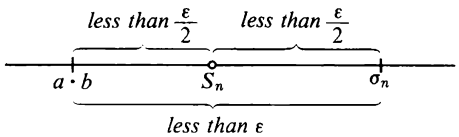


Fig. 14

(Fig. 14), that is,

$$|\sigma_n - a \cdot b| < \varepsilon$$

for sufficiently large n. As our ε is an arbitrarily small positive number, then

$$\lim_{n \to \infty} \sigma_n = a \cdot b.$$

This completes the proof that the series (***) converges and that its sum is $a \cdot b$.

As an illustration, let us consider the series

$$1 + p + p^2 + \cdots + p^n + \cdots = \frac{1}{1-p}$$

and

$$1 + q + q^2 + \cdots + q^n + \cdots = \frac{1}{1-q},$$

where $|p| < 1$ and $|q| < 1$. We know that such series (geometric progressions with the absolute values of their common ratios less than 1) converge absolutely. We form their product by the rule just established,

$$1 + (p + q) + (p^2 + pq + q^2) + (p^3 + p^2q + pq^2 + q^3) + \cdots.$$

We have proved that this series is convergent and that its sum is the product of the sums of given series. Thus

$$1 + (p + q) + (p^2 + pq + q^2) + (p^3 + p^2q + pq^2 + q^3) + \cdots = \frac{1}{(1-p)(1-q)}.$$

In particular, for $p = q$ we obtain

$$1 + 2p + 3p^2 + 4p^3 + \cdots = \frac{1}{(1-p)^2}.$$

We suggest that the reader find the products of the following series:

(*a*) $1 + 2p + 3p^2 + 4p^3 + \cdots$ and $1 + p + p^2 + p^3 + \cdots$,

(*b*) $1 + 2p + 3p^2 + 4p^3 + \cdots$ and $1 + 2p + 3p^2 + 4p^3 + \cdots$,

where $|p| < 1$.

5. The Binomial Series

33. CONDITIONS FOR CONVERGENCE OF THE BINOMIAL SERIES

Now we can return to the binomial formula, (3) of section 9,

$$(1 + x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots$$

and show that it holds not only when m is a positive integer but when m is any number. In case m is not a postive integer, the right-hand side of the equation will have no last term, as we have previously shown. It will be an infinite series (a power series), called the *binomial series.* We must first answer the question of convergence of this series. And if we establish that the series converges, we would like to verify that its sum is $(1 + x)^m$.

We shall use d'Alembert's test to establish conditions for convergence of the series (m not a positive integer)

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots.$$

In particular, we shall show that *this series converges absolutely if the absolute value of x is less than one* ($|x| < 1$), *but diverges if* $|x|$ *is greater than one.*

To prove this, we take the series consisting of the absolute values of the terms of the given series

$$1 + \left|\frac{m}{1}x\right| + \left|\frac{m(m-1)}{1\cdot 2}x^2\right| + \left|\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3\right| + \left|\frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}x^4\right| + \cdots$$

and form the ratio of each term to the one preceding it:

$$\frac{\left|\frac{m}{1}x\right|}{1} = \left|\frac{m}{1}x\right|,$$

$$\frac{\left|\frac{m(m-1)}{1\cdot 2}x^2\right|}{\left|\frac{m}{1}x\right|} = \left|\frac{m-1}{2}x\right|,$$

$$\frac{\left|\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3\right|}{\left|\frac{m(m-1)}{1\cdot 2}x^2\right|} = \left|\frac{m-2}{3}x\right|,$$

$$\frac{\left|\frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}x^4\right|}{\left|\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3\right|} = \left|\frac{m-3}{4}x\right|,$$

Here $\left|\frac{m}{1}x\right|$ is the ratio of the second term to the first one; $\left|\frac{m-1}{2}x\right|$ is the ratio of the third to the second one; $\left|\frac{m-2}{3}x\right|$ is the ratio of the fourth to the third; $\left|\frac{m-3}{4}x\right|$ is the ratio of the fifth to the fourth, and so on. Analogously, it is easy to see that the ratio of the thousandth term to the nine hundred ninety-ninth will be $\left|\frac{m-998}{999}x\right|$, and of the millionth term to the one preceding it is $\left|\frac{m-999998}{999999}x\right|$, etc.

Writing down all these quantities, starting with the second, in the form

$$\left|\left(\frac{m}{2}-\frac{1}{2}\right)x\right|,\quad \left|\left(\frac{m}{3}-\frac{2}{3}\right)x\right|,\quad \left|\left(\frac{m}{4}-\frac{3}{4}\right)x\right|,\quad \ldots,$$

$$\left|\left(\frac{m}{999}-\frac{998}{999}\right)x\right|,\quad \ldots,\quad \left|\left(\frac{m}{999999}-\frac{999998}{999999}\right)x\right|,\quad \ldots,$$

we observe that they approach the number $|x|$ as a limit. In fact, the first term in the parentheses is m, divided by larger and larger numbers, and it therefore approaches zero; the second term differs by less and less from -1 and approaches it as a limit. Therefore, the expression in the parentheses approaches -1 as a limit. Finally, x, by which we multiply each expression in parentheses, is constant from term to term. This means that the ratios approach $|-1\cdot x|$, which is just $|x|$.

Now let us assume that the absolute value of x is less than one. For clarity, we shall denote the ratios $r_1, r_2, r_3, r_4, \ldots, r_n, r_{n+1}, \ldots$ and plot them on a number line. Since these ratios approach $|x|$, then given any arbitrarily narrow interval centered about the point $|x|$, all the points r_n for n sufficiently large (say $n \geq N$) will be contained inside this interval. If we select an interval AB so narrow that it excludes the point $+1$ (Fig. 15), we may be sure that the ratios

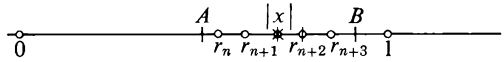


Fig. 15

corresponding to points in this interval will not be closer to one than the end point B of the interval and therefore the absolute values of the ratios r_n, r_{n+1}, r_{n+2} themselves will be less than some number q which is less than one (in Figure 15 we may take $q = OB$). Therefore the series formed by the nth term and all following terms will converge (by d'Alembert's test). But then the whole series will converge, as all its terms up to the nth form a polynomial with a definite sum, and to find the sum of the whole series it would be enough to add the sum of its first $n-1$ terms to the sum of the convergent series formed by the remaining terms.

The result will be different if the absolute value of x is greater than one. As before, given an arbitrarily small interval with its center at the point $|x|$ (Fig. 16), all the points representing ratios, starting

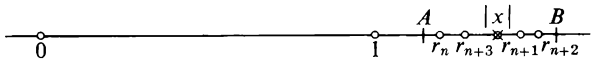


Fig. 16

with some one of them, will be contained in the interval. If we select an interval small enough for $+1$ to be outside it, we can assert that the absolute values of the ratios in the interval are greater than one and also greater than some number greater than one. And this, by d'Alembert's test, means that the series diverges.

34. PROOF OF BINOMIAL FORMULA, m ANY RATIONAL NUMBER

When $|x| < 1$, the binomial series converges and we can ask: What is its sum? We shall follow Euler's reasoning to establish that this sum equals $(1 + x)^m$. For this purpose we shall first observe that the sum of the series

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \cdots$$

is a function of both m and x, which acquires a definite numerical value for given m (arbitrary) and x ($|x|$ should be less than one; otherwise the series may diverge, as we have just seen). We shall denote this sum by $F(m, x)$ to remind ourselves that it depends upon m as well as on x. The problem is to prove that

$$F(m, x) = (1 + x)^m.$$

If this equation is true, we should have

$$\begin{aligned} F(m_1, x) \cdot F(m_2, x) &= (1 + x)^{m_1} \cdot (1 + x)^{m_2} \\ &= (1 + x)^{m_1 + m_2} \\ &= F(m_1 + m_2, x). \end{aligned}$$

We shall start with the proof of this relation; that is, we shall prove that for $|x| < 1$,

$$F(m_1, x) \cdot F(m_2, x) = F(m_1 + m_2, x),$$

or

$$\begin{aligned} &\left[1 + \frac{m_1}{1} x + \frac{m_1(m_1 - 1)}{1 \cdot 2} x^2 \right. \\ &\qquad + \frac{m_1(m_1 - 1)(m_1 - 2)}{1 \cdot 2 \cdot 3} x^3 \\ &\qquad \left. + \frac{m_1(m_1 - 1)(m_1 - 2)(m_1 - 3)}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \cdots \right] \\ &\times \left[1 + \frac{m_2}{1} x + \frac{m_2(m_2 - 1)}{1 \cdot 2} x^2 \right. \\ &\qquad + \frac{m_2(m_2 - 1)(m_2 - 2)}{1 \cdot 2 \cdot 3} x^3 \\ &\qquad \left. + \frac{m_2(m_2 - 1)(m_2 - 2)(m_2 - 3)}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \cdots \right] \\ &= 1 + \frac{m_1 + m_2}{1} x + \frac{(m_1 + m_2)(m_1 + m_2 - 1)}{1 \cdot 2} x^2 \\ &\quad + \frac{(m_1 + m_2)(m_1 + m_2 - 1)(m_1 + m_2 - 2)}{1 \cdot 2 \cdot 3} x^3 \\ &\quad + \frac{(m_1 + m_2)(m_1 + m_2 - 1)(m_1 + m_2 - 2)(m_1 + m_2 - 3)}{1 \cdot 2 \cdot 3 \cdot 4} x^4 \\ &\quad + \cdots . \end{aligned}$$

To compute this, we multiply the binomial series

$$1 + \frac{m_1}{1}x + \frac{m_1(m_1-1)}{1\cdot 2}x^2 + \frac{m_1(m_1-1)(m_1-2)}{1\cdot 2\cdot 3}x^3 + \frac{m_1(m_1-1)(m_1-2)(m_1-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots$$

by the series

$$1 + \frac{m_2}{1}x + \frac{m_2(m_2-1)}{1\cdot 2}x^2 + \frac{m_2(m_2-1)(m_2-2)}{1\cdot 2\cdot 3}x^3 + \frac{m_2(m_2-1)(m_2-2)(m_2-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots$$

term by term, that is, multiply each term of the first series by each term of the second and arrange the products in the order shown in section 32. This is legitimate because both series are absolutely convergent. We obtain the series

$$\left.\begin{aligned}
&1 + \left[\frac{m_1}{1}x + \frac{m_2}{1}x\right] \\
&+ \left[\frac{m_1(m_1-1)}{1\cdot 2}x^2 + \frac{m_1}{1}x\frac{m_2}{1}x + \frac{m_2(m_2-1)}{1\cdot 2}x^2\right] \\
&+ \left[\frac{m_1(m_1-1)(m_1-2)}{1\cdot 2\cdot 3}x^3 + \frac{m_1(m_1-1)}{1\cdot 2}x^2\cdot\frac{m_2}{1}x\right. \\
&\quad \left. + \frac{m_1}{1}x\cdot\frac{m_2(m_2-1)}{1\cdot 2}x^2 + \frac{m_2(m_2-1)(m_2-2)}{1\cdot 2\cdot 3}x^3\right] \\
&+ \left[\frac{m_1(m_1-1)(m_1-2)(m_1-3)}{1\cdot 2\cdot 3\cdot 4}x^4\right. \\
&\quad + \frac{m_1(m_1-1)(m_1-2)}{1\cdot 2\cdot 3}x^3\cdot\frac{m_2}{1}x \\
&\quad\quad + \frac{m_1(m_1-1)}{1\cdot 2}x^2\cdot\frac{m_2(m_2-1)}{1\cdot 2}x^2 \\
&\quad\quad\quad + \frac{m_1}{1}x\cdot\frac{m_2(m_2-1)(m_2-2)}{1\cdot 2\cdot 3}x^3 \\
&\quad\quad\quad\quad \left. + \frac{m_2(m_2-1)(m_2-2)(m_2-3)}{1\cdot 2\cdot 3\cdot 4}x^4\right] + \cdots,
\end{aligned}\right\} \quad (a)$$

whose sum is equal to $F(m_1, x)F(m_2, x)$. It is evident that in each bracket we can factor out a definite power of x, that is, x, x^2, x^3, x^4, ..., and then the sums of the numbers remaining in the brackets

will give the coefficients of the powers of x. We denote these coefficients in order by $B_1, B_2, B_3, B_4, \ldots$. Thus, series (a) becomes

$$1 + B_1x + B_2x^2 + B_3x^3 + B_4x^4 + \cdots = F(m_1, x) \cdot F(m_2, x)$$

where the coefficients have the following values:

$$B_1 = m_1 + m_2,$$

$$B_2 = \frac{m_1(m_1 - 1)}{1 \cdot 2} + \frac{m_1}{1} \cdot \frac{m_2}{1} + \frac{m_2(m_2 - 1)}{1 \cdot 2},$$

$$B_3 = \frac{m_1(m_1 - 1)(m_1 - 2)}{1 \cdot 2 \cdot 3} + \frac{m_1(m_1 - 1)}{1 \cdot 2} \cdot \frac{m_2}{1} + \frac{m_1}{1} \cdot \frac{m_2(m_2 - 1)}{1 \cdot 2} + \frac{m_2(m_2 - 1)(m_2 - 2)}{1 \cdot 2 \cdot 3},$$

$$B_4 = \frac{m_1(m_1 - 1)(m_1 - 2)(m_1 - 3)}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{m_1(m_1 - 1)(m_1 - 2)}{1 \cdot 2 \cdot 3} \cdot \frac{m_2}{1} + \frac{m_1(m_1 - 1)}{1 \cdot 2} \cdot \frac{m_2(m_2 - 1)}{1 \cdot 2} + \frac{m_1}{1} \cdot \frac{m_2(m_2 - 1)(m_2 - 2)}{1 \cdot 2 \cdot 3} + \frac{m_2(m_2 - 1)(m_2 - 2)(m_2 - 3)}{1 \cdot 2 \cdot 3 \cdot 4},$$

We must now perform some computations to show that these coefficients may be written in the following form:

$$B_1 = \frac{m_1 + m_2}{1},$$

$$B_2 = \frac{(m_1 + m_2)(m_1 + m_2 - 1)}{1 \cdot 2},$$

$$B_3 = \frac{(m_1 + m_2)(m_1 + m_2 - 1)(m_1 + m_2 - 2)}{1 \cdot 2 \cdot 3},$$

$$B_4 = \frac{(m_1 + m_2)(m_1 + m_2 - 1)(m_1 + m_2 - 2)(m_1 + m_2 - 3)}{1 \cdot 2 \cdot 3 \cdot 4},$$

From this it will follow that the sum of the series (a) coincides with $F(m_1 + m_2, x)$; that is,

$$F(m_1, x) \cdot F(m_2, x) = F(m_1 + m_2, x).$$

The formula for B_1 needs no proof. Before proving the other formulas, we shall write them with the known expressions for the

coefficients B_2, B_3, B_4, . . . on the left-hand side and multiply both sides of the formula for B_2 by $1 \cdot 2$, of the formula for B_3 by $1 \cdot 2 \cdot 3$, of the formula for B_4 by $1 \cdot 2 \cdot 3 \cdot 4$, etc. Then the formulas we wish to prove will take the form

$$m_1(m_1 - 1) + 2m_1m_2 + m_2(m_2 - 1) = (m_1 + m_2)(m_1 + m_2 - 1), \qquad (b)$$

$$m_1(m_1 - 1)(m_1 - 2) + 3m_1(m_1 - 1)m_2 + 3m_1m_2(m_2 - 1) + m_2(m_2 - 1)(m_2 - 2) = (m_1 + m_2)(m_1 + m_2 - 1)(m_1 + m_2 - 2), \qquad (c)$$

$$m_1(m_1 - 1)(m_1 - 2)(m_1 - 3) + 4m_1(m_1 - 1)(m_1 - 2)m_2 + 6m_1(m_1 - 1)m_2(m_2 - 1) + 4m_1m_2(m_2 - 1)(m_2 - 2) + m_2(m_2 - 1)(m_2 - 2)(m_2 - 3) = (m_1 + m_2)(m_1 + m_2 - 1)(m_1 + m_2 - 2)(m_1 + m_2 - 3), \qquad (d)$$

. . . . . .

These identities display some similarity to the known identities

$$m_1 \cdot m_1 + 2m_1 \cdot m_2 + m_2 \cdot m_2 = (m_1 + m_2)(m_1 + m_2),$$

$$m_1 \cdot m_1 \cdot m_1 + 3m_1 \cdot m_1 \cdot m_2 + 3m_1 \cdot m_2 \cdot m_2 + m_2 \cdot m_2 \cdot m_2 = (m_1 + m_2)(m_1 + m_2)(m_1 + m_2),$$

$$m_1 \cdot m_1 \cdot m_1 \cdot m_1 + 4m_1m_1m_1m_2 + 6m_1m_1m_2m_2 + 4m_1m_2m_2m_2 + m_2m_2m_2m_2 = (m_1 + m_2)(m_1 + m_2)(m_1 + m_2)(m_1 + m_2),$$

. . . . . .

To prove formulas (b), (c), (d) . . . , we shall derive first the formula (b) from the identity $m_1 + m_2 = m_1 + m_2$, then the formula (c) from (b), then the formula (d) from (c), and so on. To obtain the right-hand side of (b) we multiply both sides of the identity $m_1 + m_2 = m_1 + m_2$ by $m_1 + m_2 - 1$. We now show that after this we can transform the left-hand side into the form of the left-hand side of (b):

$$(m_1 + m_2)(m_1 + m_2 - 1) = m_1[(m_1 - 1) + m_2] + m_2[m_1 + (m_2 - 1)]$$

$$= m_1(m_1 - 1) + m_1m_2 + m_2m_1 + m_2(m_2 - 1)$$
$$= m_1(m_1 - 1) + 2m_1m_2 + m_2(m_2 - 1).$$

Thus we have proved identity (*b*). In order to obtain from its right-hand side the right-hand side of (*c*), we multiply each term of (*b*) by $m_1 + m_2 - 2$. Then we can transform the left-hand side into the form of the left-hand side of (*c*):

$$\begin{aligned}
&[m_1(m_1 - 1) + 2m_1m_2 + m_2(m_2 - 1)](m_1 + m_2 - 2)\\
&= m_1(m_1 - 1)[(m_1 - 2) + m_2] + 2m_1m_2[(m_1 - 1)\\
&\qquad + (m_2 - 1)] + m_2(m_2 - 1)[m_1 + (m_2 - 2)]\\
&= m_1(m_1 - 1)(m_1 - 2) + m_1(m_1 - 1)m_2 + 2m_1(m_1 - 1)m_2\\
&\qquad + 2m_1m_2(m_2 - 1) + m_1m_2(m_2 - 1) + m_2(m_2 - 1)(m_2 - 2)\\
&= m_1(m_1 - 1)(m_1 - 2) + 3m_1(m_1 - 1)m_2 + 3m_1m_2(m_2 - 1)\\
&\qquad + m_2(m_2 - 1)(m_2 - 2).
\end{aligned}$$

Thus we have proved identity (*c*). In order to obtain from its right-hand side the right-hand side of (*d*), we multiply each term of (*c*) by $m_1 + m_2 - 3$. Then we can transform the left-hand side into the form of the left-hand side of (*d*):

$$\begin{aligned}
&[m_1(m_1 - 1)(m_1 - 2) + 3m_1(m_1 - 1)m_2 + 3m_1m_2(m_2 - 1)\\
&\quad + m_2(m_2 - 1)(m_2 - 2)](m_1 + m_2 - 3)\\
&= m_1(m_1 - 1)(m_1 - 2)[(m_1 - 3) + m_2] + 3m_1(m_1 - 1)m_2[(m_1\\
&\quad -2) + (m_2 - 1)] + 3m_1m_2(m_2 - 1)[(m_1 - 1) + (m_2 - 2)]\\
&\quad + m_2(m_2 - 1)(m_2 - 2)[m_1 + (m_2 - 3)]\\
&= m_1(m_1 - 1)(m_1 - 2)(m_1 - 3) + m_1(m_1 - 1)(m_1 - 2)m_2\\
&\quad + 3m_1(m_1 - 1)(m_1 - 2)m_2 + 3m_1(m_1 - 1)m_2(m_2 - 1)\\
&\quad + 3m_1(m_1 - 1)m_2(m_2 - 1) + 3m_1m_2(m_2 - 1)(m_2 - 2)\\
&\quad + m_1m_2(m_2 - 1)(m_2 - 2) + m_2(m_2 - 1)(m_2 - 2)(m_2 - 3)\\
&= m_1(m_1 - 1)(m_1 - 2)(m_1 - 3) + 4m_1(m_1 - 1)(m_1 - 2)m_2\\
&\quad + 6m_1(m_1 - 1)m_2(m_2 - 1) + 4m_1m_2(m_2 - 1)(m_2 - 2)\\
&\quad + m_2(m_2 - 1)(m_2 - 2)(m_2 - 3).
\end{aligned}$$

Thus we have proved identity (*d*). We suggest that the reader derive from (*d*) a new identity by multiplying both sides by

$m_1 + m_2 - 4$. After performing the necessary multiplications on the left-hand side, we shall need only to group like terms. We shall then have to add pairs of the binomial coefficients of the fourth power: $1 + 4$, $4 + 6$, $6 + 4$, $4 + 1$. Of course, we should obtain, as coefficients on the left-hand side, the coefficients of the fifth power. Finally, we come to a new identity:

$$\begin{aligned} &m_1(m_1 - 1)(m_1 - 2)(m_1 - 3)(m_1 - 4) + 5m_1(m_1 - 1)(m_1 \\ &\quad - 3)m_2 + 10m_1(m_1 - 1)(m_1 - 2)m_2(m_2 - 1) + 10m_1(m_1 \\ &\quad - 1)m_2(m_2 - 1)(m_2 - 2) + 5m_1m_2(m_2 - 1)(m_2 - 2)(m_2 - 3) \\ &\quad + m_2(m_2 - 1)(m_2 - 2)(m_2 - 3)(m_2 - 4) \\ &= (m_1 + m_2)(m_1 + m_2 - 1)(m_1 + m_2 - 2)(m_1 \\ &\quad + m_2 - 3)(m_1 + m_2 - 4). \end{aligned}$$

It is evident that multiplying both sides by $m_1 + m_2 - 5$ and transforming the left-hand side as we have done previously, we obtain an identity in which the coefficients on the left-hand side will coincide with the binomial coefficients of the sixth degree, etc. From the identities proved above it follows that the coefficients of the series (*a*) (p. 96) coincide with the corresponding coefficients of the series $F(m_1 + m_2, x)$.

We have proved that

$$F(m_1, x) \cdot F(m_2, x) = F(m_1 + m_2, x) \qquad (e)$$

for any m_1 and m_2 and for $|x| < 1$. But what does this mean? At present only that the sum of the binomial series $F(m, x)$ possesses a property of the polynomial $(1 + x)^m$: as in the multiplication of two polynomials $(1 + x)^{m_1}$ and $(1 + x)^{m_2}$ we add the exponents m_1 and m_2 and obtain $(1 + x)^{m_1+m_2}$, so in the multiplication of two binomial series $F(m_1, x)$ and $F(m_2, x)$ we add the numbers m_1 and m_2 and obtain the binomial series $F(m_1 + m_2, x)$.

May we conclude from this that the sum of the binomial series

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots = F(m, x)$$

is equal to $(1 + x)^m$? We shall show now that we may.

For this purpose, we observe first that

$$F(0, x) = 1 + \frac{0}{1}x + \frac{0(0-1)}{1 \cdot 2}x^2 + \frac{0(0-1)(0-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots = 1$$

and

$$F(1, x) = 1 + \frac{1}{1}x + \frac{1(1-1)}{1 \cdot 2}x^2 + \frac{1(1-1)(1-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$
$$= 1 + x.$$

Further, $F(2, x) = F(1, x) \cdot F(1, x)$ because of equation (e). Consequently,

$$F(2, x) = [F(1, x)]^2 = (1 + x)^2.$$

We also obtain from equation (e):

$$F(3, x) = F(2, x) \cdot F(1, x) = (1 + x)^2 \cdot (1 + x) = (1 + x)^3,$$
$$F(4, x) = F(3, x) \cdot F(1, x) = (1 + x)^3 \cdot (1 + x) = (1 + x)^4, \text{ etc.}$$

By induction we can easily prove that

$$F(m, x) = (1 + x)^m$$

for any positive integer m. This gives us nothing new, since the binomial formula was already established in Chapter 2 for positive integral exponents.[1]

We shall now consider the case of a negative integer $-m$. By equation (e), we have

$$F(-m, x) \cdot F(m, x) = F(-m + m, x) = F(0, x).$$

Now m is a positive integer, so that, as we have proved,

$$F(m, x) = (1 + x)^m. \qquad (f)$$

Also, we have seen that

$$F(0, x) = 1.$$

Therefore,

$$F(-m, x) \cdot (1 + x)^m = 1$$

and

$$F(-m, x) = \frac{1}{(1 + x)^m} = (1 + x)^{-m}. \qquad (g)$$

Thus the binomial formula holds also for negative integral exponents.

[1] For a positive integer m we do not need the restriction $|x| < 1$, which in general is necessary for convergence of the series, as here the series turns into a polynomial (all terms of the series containing x to a power greater than m have coefficients equal to zero).

Now we set $m = \frac{p}{q}$ where p is any integer and q is a positive integer. Then

$$F\left(\frac{p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right) = F\left(\frac{p}{q} + \frac{p}{q}, x\right) = F\left(\frac{2p}{q}, x\right),$$

$$\begin{aligned} F\left(\frac{p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right) &= \left[F\left(\frac{p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right)\right] \cdot F\left(\frac{p}{q}, x\right) \\ &= F\left(\frac{2p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right) = F\left(\frac{2p}{q} + \frac{p}{q}, x\right) = F\left(\frac{3p}{q}, x\right), \end{aligned}$$

$$\begin{aligned} F\left(\frac{p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right) &= \left[F\left(\frac{p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right)\right] \cdot F\left(\frac{p}{q}, x\right) \\ &= F\left(\frac{3p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right) = F\left(\frac{3p}{q} + \frac{p}{q}, x\right) = F\left(\frac{4p}{q}, x\right), \end{aligned}$$

etc.

Similarly, multiplying $F\left(\frac{p}{q}, x\right)$ by itself q times, we obtain

$$\begin{aligned} \left[F\left(\frac{p}{q}, x\right)\right]^q &= \underbrace{F\left(\frac{p}{q}, x\right) \cdot F\left(\frac{p}{q}, x\right) \cdots F\left(\frac{p}{q}, x\right)}_{q \text{ factors}} \\ &= F\left(\frac{qp}{q}, x\right) = F(p, x). \end{aligned}$$

But p is an integer; consequently,

$$F(p, x) = (1 + x)^p,$$

and we have

$$\left[F\left(\frac{p}{q}, x\right)\right]^q = (1 + x)^p.$$

From this we would like to conclude that

$$F\left(\frac{p}{q}, x\right) = \sqrt[q]{(1+x)^p} = (1+x)^{\frac{p}{q}} \qquad (h)$$

by taking the qth root of both sides of the previous equation. But there are complications. If q is odd, a number will have exactly one real qth root, but if q is even, a positive number will have two real qth roots (they will have equal magnitudes and opposite signs: for example, $2^2 = (-2)^2 = 4$. So, when q is even, we must discover whether $F\left(\frac{p}{q}, x\right)$ is positive or negative. Let us remark that the notations used in formula (h) (radical sign and fractional exponent) are used by present-day mathematicians to indicate only the positive root, not the negative. We shall show that $F(m, x)$ is always positive and hence that formula (h) is correct as it stands.

From the basic property (e) of the function $F(m, x)$, we obtain

$$\begin{aligned} F(m, x) &= F\left(\frac{m}{2}, x\right) \cdot F\left(\frac{m}{2}, x\right) \\ &= \left[F\left(\frac{m}{2}, x\right)\right]^2. \end{aligned}$$

Therefore, $F(m, x)$ is positive, since it is equal to the square of a real number. Thus (h) is correct and the formula

$$F(m, x) = (1+x)^m,$$

that is, the binomial formula

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots = (1+x)^m$$

holds for any m, positive or negative, integer or fraction. (If m is not a positive integer in this formula, then we must use only x with absolute value less than one; otherwise the binomial series would diverge.)

35. PROOF OF BINOMIAL FORMULA FOR m AN IRRATIONAL NUMBER

We have not yet completely solved the problem, since besides integers and fractions there exist irrational numbers, which are neither integers nor fractions. (For instance, $\sqrt{2}$ is not an integer and at the same time is not a fraction, as there is no fraction $\frac{p}{q}$ with integral numerator and denominator whose square is 2.) Now we must recall the meaning of a power with an irrational exponent. What, for instance, does $1.1^{\sqrt{2}}$ mean?

To answer this question, we shall look for rational numbers differing less and less from the given irrational number α (the exponent). Thus for $\sqrt{2}$ we can take as such numbers 1.4 (the value of $\sqrt{2}$ accurate within 0.01), 1.414 (within 0.001), 1.4142 (within 0.0001), etc. Now, instead of considering the power with the irrational exponent, we shall consider powers with these rational exponents, each of which will have a definite algebraic meaning. In our example, instead of $1.1^{\sqrt{2}}$, we shall consider

$$1.1^{1.4} = \sqrt[10]{1.1^{14}} = \sqrt[5]{1.1^{7}},$$
$$1.1^{1.41} = \sqrt[100]{1.1^{141}},$$
$$1.1^{1.414} = \sqrt[1000]{1.1^{1414}} = \sqrt[500]{1.1^{707}},$$
$$1.1^{1.4142} = \sqrt[10000]{1.1^{14142}} = \sqrt[5000]{1.1^{7071}}, \text{ etc.}$$

We can prove that as the rational exponents $r_1, r_2, r_3, r_4, \ldots$ (in our case $r_1 = 1.4$, $r_2 = 1.41$, $r_3 = 1.414$, $r_4 = 1.4142, \ldots$) approach the irrational number α (here, $\alpha = \sqrt{2}$) as a limit, the corresponding powers $a^{r_1}, a^{r_2}, a^{r_3}, a^{r_4}, \ldots$ (in our example $1.1^{1.4}$, $1.1^{1.41}$, $1.1^{1.414}$, $1.1^{1.4142}, \ldots$) also approach some limit A. We consider this limit as the power of the number a with irrational exponent α; that is, a^{α} is defined to be A. In other words, by a power of any positive number a with an irrational exponent α we mean the limit of powers of a with rational exponents under the condition that these exponents approach the number α. This is expressed by the formula

$$a^{\alpha} = \lim_{r_n \to \alpha} a^{r_n}.$$

Returning to the example, we should consider powers with rational exponents

$$1.1^{1.4}, 1.1^{1.41}, 1.1^{1.414}, 1.1^{1.4142}, \ldots$$

as (more and more accurate) approximate values of the desired number $1.1^{\sqrt{2}}$. In order to compute these approximate values, we will not extract roots with indices 5, 100, 500, 5000, . . . (we have seen previously that $1.1^{1.4} = \sqrt[5]{1.1^7}$, $1.1^{1.41} = \sqrt[10]{1.1^{141}}$, etc.), because we simply do not know how to perform this, but we shall turn to the binomial formula. Its use is completely legitimate here since we have already proved it for rational exponents.

In our example, we have

$$1.1^{1.4} = (1 + 0.1)^{1.4}, \qquad 1.1^{1.41} = (1 + 0.1)^{1.41}, \text{ etc.}$$

($x = 0.1 < 1$). The binomial formula yields

$$1.1^{1.4} = (1 + 0.1)^{1.4} = 1 + \frac{1.4}{1} \cdot 0.1 + \frac{1.4(1.4 - 1)}{1 \cdot 2} \cdot (0.1)^2 + \cdots$$
$$= 1 + 0.14 + 0.0028 + \cdots \approx 1.1428,$$

$$1.1^{1.41} = (1 + 0.1)^{1.41} = 1 + \frac{1.41}{1} \cdot 0.1 + \frac{1.41(1.41 - 1)}{1 \cdot 2} \cdot (0.1)^2 + \cdots$$
$$= 1 + 0.141 + 0.0029 + \cdots \approx 1.1439,$$

$$1.1^{1.414} = (1 + 0.1)^{1.414}$$
$$= 1 + \frac{1.414}{1} \cdot 0.1 + \frac{1.414(1.414 - 1)}{1 \cdot 2} \cdot (0.1)^2 + \cdots$$
$$= 1 + 0.1414 + 0.0029 + \cdots \approx 1.1443,$$

$$1.1^{1.4142} = (1 + 0.1)^{1.4142}$$
$$= 1 + \frac{1.4142}{1} \cdot 0.1 + \frac{1.4142(1.4142 - 1)}{1 \cdot 2} \cdot (0.1)^2$$
$$+ \frac{1.4142(1.4142 - 1)(1.4142 - 2)}{1 \cdot 2 \cdot 3} \cdot (0.1)^3 + \cdots$$
$$= 1 + 0.14142 + 0.00293 - 0.00006 + \cdots \approx 1.14429.$$

In the last computation, we took not three but four terms of the binomial formula because the omission of the fourth term (equal to -0.00006) would cause an error that would make the great accuracy of the exponent useless. The last result obtained gives $1.1^{\sqrt{2}}$ accurate within 0.00001. If we use more accurate approximations of $\sqrt{2}$ as exponents, then we obtain results in which the first six digits are the same as in the number 1.14429 we have just computed.

After this digression, we can prove without difficulty that the binomial formula also holds for irrational exponents. To begin, let m be an irrational number and let $|x| < 1$. We shall prove that

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \cdots = (1+x)^m.$$

For this purpose, we shall consider a sequence of rational numbers $r_1, r_2, r_3, r_4, \ldots$ which approach the number m as a limit:

$$r_n \to m.$$

If in the binomial theorem we substitute the numbers $r_1, r_2, r_3, r_4, \ldots$ for m, then we obtain the series

$$1 + \frac{r_1}{1}x + \frac{r_1(r_1-1)}{1\cdot 2}x^2 + \frac{r_1(r_1-1)(r_1-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

$$1 + \frac{r_2}{1}x + \frac{r_2(r_2-1)}{1\cdot 2}x^2 + \frac{r_2(r_2-1)(r_2-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

$$1 + \frac{r_3}{1}x + \frac{r_3(r_3-1)}{1\cdot 2}x^2 + \frac{r_3(r_3-1)(r_3-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

$$1 + \frac{r_4}{1}x + \frac{r_4(r_4-1)}{1\cdot 2}x^2 + \frac{r_4(r_4-1)(r_4-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

whose terms approach the terms of the series

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \cdots.$$

From this alone we cannot conclude that the sums of our series approach the sum of the last series. We omit the proof, which is not very difficult, that this does occur in our case; that is, that

$$\lim_{r_n \to m}\left[1 + \frac{r_n}{1}x + \frac{r_n(r_n-1)}{1\cdot 2}x^2 + \frac{r_n(r_n-1)(r_n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\right]$$
$$= 1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \cdots.$$

The sum of the series

$$1 + \frac{r_n}{1}x + \frac{r_n(r_n-1)}{1\cdot 2}x^2 + \frac{r_n(r_n-1)(r_n-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

is $(1 + x)^{r_n}$ (r_n is a rational number, and we have already proved the binomial formula for rational exponents). Therefore we have:

$$\lim_{r_n \to m} (1 + x)^{r_n} = 1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$

and since $\lim_{r_n \to m} (1 + x)^{r_n}$ equals $(1 + x)^m$ by the definition of a power with an irrational exponent, we finally obtain

$$(1 + x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots,$$

which is what we had to prove.

In particular, we have

$$1.1^{\sqrt{2}} = (1 + 0.1)^{\sqrt{2}} = 1 + \frac{\sqrt{2}}{1} \cdot 0.1 + \frac{\sqrt{2}(\sqrt{2}-1)}{1 \cdot 2} \cdot (0.1)^2 + \frac{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}-2)}{1 \cdot 2 \cdot 3} \cdot (0.1)^3 + \cdots.$$

If we wanted to use this formula to compute $1.1^{\sqrt{2}}$ we would limit ourselves to the first few terms of the formula (three, four, five) and replace $\sqrt{2}$ by an approximate value (such as 1.41, 1.4142). In this way, we would repeat all the above computations (on p. 105).

We have completed the proof of Newton's formula, that is, we have proved that

$$(1 + x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$

for any real number m and $|x| < 1$.

It may be useful to observe that Newton himself had no proof and that the proof was made possible by the work of several prominent mathematicians (Newton, Euler, Cauchy, Abel, and others) during a century and a half. The reason for the delay was not so much the technical difficulties of the proof (which are minor) as the development of clear and accurate concepts connected with infinite series and operations on them. This in turn required clarification of the concept of a limit and study of laws concerning passage to a limit.

36. APPLICATIONS

If we set $m = \frac{1}{2}$, $m = \frac{1}{3}$, $m = -1$, $m = -\frac{1}{2}$, $m = -\frac{1}{3}$ in succession, we obtain

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots, \qquad (\alpha)$$

$$\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$$
$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots, \qquad (\beta)$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots, \qquad (\gamma)$$

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$$
$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 - \cdots, \qquad (\delta)$$

$$\frac{1}{\sqrt[3]{1+x}} = (1+x)^{-\frac{1}{3}}$$
$$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4 - \cdots. \qquad (\varepsilon)$$

If $|x|$ is a small number, then x^2, x^3, x^4, ... are very small numbers. Therefore we shall not make a large error if we discard from each of the above formulas all the terms starting with the third (which contains x^2). We obtain the following approximate formulas:

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x, \qquad (\alpha')$$

$$\sqrt[3]{1+x} \approx 1 + \frac{1}{3}x, \qquad (\beta')$$

$$\frac{1}{1+x} \approx 1 - x, \qquad (\gamma')$$

$$\frac{1}{\sqrt{1+x}} \approx 1 - \frac{1}{2}x, \qquad (\delta')$$

$$\frac{1}{\sqrt[3]{1+x}} \approx 1 - \frac{1}{3}x. \qquad (\varepsilon')$$

It is easy to evaluate the errors caused by the use of these formulas when $x > 0$. We then obtain alternating series with decreasing terms. Therefore the error due to discarding terms of the series does not exceed the first of the discarded terms. Consequently, for $x > 0$ the error in the first formula is less than $\frac{1}{8}x^2$, in the second—less than $\frac{1}{9}x^2$, in the third—less than x^2, in the fourth—less than $\frac{3}{8}x^2 < \frac{1}{2}x^2$, finally, in the fifth—less than $\frac{2}{9}x^2 < \frac{1}{4}x^2$.

We shall give examples of computations using these formulas.

From formula (α'):

$\sqrt{1.02} \approx 1.01$, with error less than $\frac{1}{8}(0.02)^2 = 0.00005$.

$\sqrt{0.97} = \sqrt{1 - 0.03} \approx 1 - \frac{0.03}{2} = 0.985$.

From formula (β'):

$\sqrt[3]{1.07} \approx 1 + \frac{0.07}{3} \approx 1.023$, with error less than $\frac{1}{9}(0.07)^2 < 0.0006$.

$\sqrt[3]{0.82} = \sqrt[3]{1 - 0.18} \approx 1 - \frac{0.18}{3} = 0.94$.

From formula (γ'):

$\frac{1}{1.06} \approx 1 - 0.06 = 0.94$, with error less than $(0.06)^2 = 0.0036 < 0.004$.

$\frac{1}{0.95} = \frac{1}{1 - 0.05} \approx 1 + 0.05 = 1.05$.

From formula (δ'):

$\frac{1}{\sqrt{1.04}} \approx 1 - \frac{0.04}{2} = 0.98$, with error less than

$$\frac{1}{2}(0.04)^2 = 0.0008 < 0.001.$$

$\frac{1}{\sqrt{0.98}} = \frac{1}{\sqrt{1 - 0.02}} \approx 1 - \frac{-0.02}{2} = 1.01$.

From formula (ε'):

$$\frac{1}{\sqrt[3]{1.06}} \approx 1 - \frac{0.06}{3} = 0.98, \text{ with error less than}$$

$$\frac{1}{4}(0.06)^2 = 0.0009 < 0.001.$$

$$\frac{1}{\sqrt[3]{0.94}} = \frac{1}{\sqrt[3]{1 - 0.06}} \approx 1 - \frac{-0.06}{3} = 1.02.$$

In all these examples the error is small. If in the expression $(1 + x)^m$ we take a larger x, that is, tenths instead of hundredths, the error will be larger. For instance,

$\sqrt{1.24} \approx 1.12$ with an error less than $\frac{1}{8}(0.24)^2 = 0.007 \cdots < 0.01$;

$\sqrt{1.42} \approx 1.21$ with an error less than $\frac{1}{8}(0.42)^2 \approx 0.02$.

Here we cannot even guarantee the second digit after the decimal point. If we wish to apply formula (α') to the case

$$\sqrt{1.9} \approx 1 + \frac{0.9}{2} = 1.45,$$

then the error will be on the order of $\frac{1}{8}(0.9)^2 \approx 0.1$ (actually $\sqrt{1.9} = 1.378\ldots$).

We should not think that formulas (α') through (ε') can be used only with numbers differing little from one, that is, numbers of the form $(1 + x)$ where the absolute value of x is small. We shall show, for instance, how we can use formulas (α') and (β') to compute square and cube roots of arbitrary numbers:

$$\sqrt{156} = \sqrt{1.56 \cdot 100} = 10\sqrt{1.56}$$

$$\approx 10\left(1 + \frac{0.56}{2}\right) = 10 \cdot 1.28 = 12.8.$$

The error in this case is less than $10 \cdot \frac{1}{8}(0.56)^2 \approx 0.4$.

This error represents $\frac{0.4 \cdot 100}{12.8}$ % of our result, or about 3%, which is a significant error. We can decrease it if we observe that 156 is fairly close to $12^2 = 144$. We can then write

$$\sqrt{156} = \sqrt{144 + 12} = \sqrt{144\left(1 + \frac{12}{144}\right)};$$

thus,

$$\sqrt{156} = 12\sqrt{1 + \frac{1}{12}} \approx 12\left(1 + \frac{1}{24}\right) = 12.5.$$

Here the error is no greater than $12 \cdot \frac{1}{8} \cdot \frac{1}{12^2} = \frac{1}{96} \approx 0.01$.

We shall now find $\sqrt{175}$, observing that 175 is close to $169 = 13^2$. We obtain

$$\sqrt{175} = \sqrt{169 + 6} = \sqrt{169\left(1 + \frac{6}{169}\right)}$$

$$= 13\sqrt{1 + \frac{6}{169}} \approx 13\left(1 + \frac{3}{169}\right) \approx 13.23.$$

The error is this result is less than $13 \cdot \frac{1}{8} \cdot \left(\frac{6}{169}\right)^2 < 0.002$.

Finally, let us compute $\sqrt[3]{500}$. Since 500 is close to $512 = 8^3$, we write

$$\sqrt[3]{500} = \sqrt[3]{512 - 12} = \sqrt[3]{512\left(1 - \frac{12}{512}\right)}$$

$$= 8\sqrt[3]{1 - \frac{3}{128}} \approx 8\left(1 - \frac{1}{128}\right) = 8 - \frac{1}{16} \approx 7.938.$$

The error in this result is less than 0.002.

In general, the nearer the radicand is to a perfect square or cube[1] the more accurate are the results obtained by this method of approximation of roots.

[1] More precisely, if $A = a^2 + b$, where $|b| < a^2$, then

$$\sqrt{A} = \sqrt{a^2 + b} = \sqrt{a^2\left(1 + \frac{b}{a^2}\right)} = a\sqrt{1 + \frac{b}{a^2}} \approx a\left(1 + \frac{b}{2a^2}\right) = a + \frac{b}{2a}.$$

The error in the result for $\frac{b}{a^2} > 0$ is no greater than

$$a \cdot \frac{1}{8}\left(\frac{b}{a^2}\right)^2 = \frac{b^2}{8a^3}.$$

Analogously, for $A = a^3 + b$, where $|b| < |a^3|$, we have

$$\sqrt[3]{A} = \sqrt[3]{a^3 + b} = \sqrt[3]{a^3\left(1 + \frac{b}{a^3}\right)} = a\sqrt[3]{1 + \frac{b}{a^3}} \approx a\left(1 + \frac{b}{3a^3}\right) = a + \frac{b}{3a^2}.$$

For $\frac{b}{a^3} > 0$ the error in this formula is less than $a \cdot \frac{1}{9}\left(\frac{b}{a^3}\right)^3 = \frac{b^3}{9a^8}$.

6. Series Expansions for the Sine and Cosine

In the preceding chapters we expanded the function $(1 + x)^n$ into an infinite series, and we mentioned the significance of such expansions. Now we shall find series expansions for the trigonometric functions $\sin x$ and $\cos x$.

37. MULTIPLE ANGLE FORMULAS

First, let us recall some basic facts. The trigonometric functions $\sin x$ and $\cos x$ of the angle x have a simple geometric interpretation. Let us draw a circle with radius 1 (Fig. 17) and select one of the radii, for instance OA, as the initial radius. If we construct an angle AOB with its vertex at the center O of the circle and points A, B on the circle, then the length of the perpendicular BC, dropped from B to OA, gives the sine of the angle AOB, and the length of the segment OC gives the cosine. Here the measure of the angle AOB, called its *radian* measure, is defined to be the length of the arc AB. Any positive or negative number can represent an angle: The number is positive if the angle is measured counterclockwise around the circle, and it is negative if measured clockwise. The absolute value of x may be greater than 2π (that is, greater than the circumference), and in this case it is useful to imagine that the angle AOB is formed by rotating the terminal radius OB from the initial position OA. For instance, to obtain the angle 3π we rotate OB counterclockwise by one whole revolution (2π) and a half (π), so that its final position will be OB_1. It appears in the same position as the angle π or (180°). But considering the way in which we formed this angle, we nevertheless call it a 3π angle. We should also mention that for various positions of the terminal radius OB the sine line, BC, may be found either above or

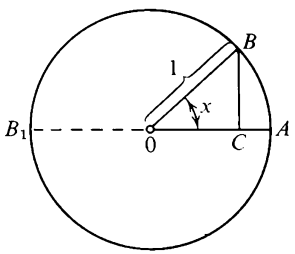


Fig. 17

below the fixed radius OA, the cosine line can be either to the left or to the right of the center: these cases determine the sign of the trigonometric functions. The sine is positive if the sine line lies above the fixed diameter and negative if it lies below. The cosine is positive if the cosine line lies to the right of the center of the circle and negative if it lies to the left.

Here we need only the two most important trigonometric formulas: the multiple angle formulas,

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \sin\beta\cos\alpha, \tag{1a}$$

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha\sin\beta. \tag{1b}$$

We shall use them as the basis of our expansion of trigonometric functions into power series. First, we shall derive formulas that will enable us to find the sine and cosine of double, triple, quadruple, and other multiple angles from the sine and cosine of a single angle.

It is easy to derive the double angle formulas for the sine and cosine from formulas (1a) and (1b):

$$\begin{aligned}\sin 2x &= \sin(x + x)\\ &= \sin x\cos x + \sin x\cos x = 2\sin x\cos x,\end{aligned} \tag{2a}$$

$$\begin{aligned}\cos 2x &= \cos(x + x)\\ &= \cos x\cos x - \sin x\sin x = \cos^2 x - \sin^2 x.\end{aligned} \tag{2b}$$

Now we can obtain the sine and cosine of triple angles:

$$\begin{aligned}\sin 3x &= \sin(2x + x)\\ &= \sin 2x\cos x + \sin x\cos 2x\\ &= (2\cos x\sin x)\cos x + \sin x(\cos^2 x - \sin^2 x)\\ &= 2\cos^2 x\sin x + \cos^2 x\sin x - \sin^3 x\\ &= 3\cos^2 x\sin x - \sin^3 x,\end{aligned} \tag{3a}$$

$$\begin{aligned}\cos 3x &= \cos(2x + x)\\ &= \cos 2x\cos x - \sin 2x\sin x\\ &= (\cos^2 x - \sin^2 x)\cos x - (2\cos x\sin x)\sin x\\ &= \cos^3 x - \cos x\sin^2 x - 2\cos x\sin^2 x\\ &= \cos^3 x - 3\cos x\sin^2 x.\end{aligned} \tag{3b}$$

We shall also derive formulas for the sine and cosine of fourfold and fivefold angles:

$$\begin{aligned}\sin 4x &= \sin(3x + x)\\ &= \sin 3x \cos x + \sin x \cos 3x\\ &= (3\cos^2 x \sin x - \sin^3 x)\cos x\\ &\qquad + \sin x\,(\cos^3 x - 3\cos x \sin^2 x)\\ &= 3\cos^3 x \sin x - \cos x \sin^3 x\\ &\qquad + \cos^3 x \sin x - 3\cos x \sin^3 x\\ &= 4\cos^3 x \sin x - 4\cos x \sin^3 x, \qquad (4a)\end{aligned}$$

$$\begin{aligned}\cos 4x &= \cos(3x + x)\\ &= \cos 3x \cos x - \sin 3x \sin x\\ &= (\cos^3 x - 3\cos x \sin^2 x)\cos x\\ &\qquad - (3\cos^2 x \sin x - \sin^3 x)\sin x\\ &= \cos^4 x - 3\cos^2 x \sin^2 x - 3\cos^2 x \sin^2 x + \sin^4 x\\ &= \cos^4 x - 6\cos^2 x \sin^2 x + \sin^4 x, \qquad (4b)\end{aligned}$$

$$\begin{aligned}\sin 5x &= \sin(4x + x)\\ &= \sin 4x \cos x + \cos 4x \sin x\\ &= (4\cos^3 x \sin x - 4\cos x \sin^3 x)\cos x\\ &\qquad + \sin x\,(\cos^4 x - 6\cos^2 x \sin^2 x + \sin^4 x\\ &= 4\cos^4 x \sin x - 4\cos^2 x \sin^3 x + \cos^4 x \sin x\\ &\qquad - 6\cos^2 x \sin 3x + \sin^5 x\\ &= 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x, \qquad (5a)\end{aligned}$$

$$\begin{aligned}\cos 5x &= \cos(4x + x)\\ &= \cos 4x \cos x - \sin 4x \sin x\\ &= (\cos^4 x - 6\cos^2 x \sin^2 x + \sin^4 x)\cos x\\ &\qquad - (4\cos^3 x \sin x - 4\cos x \sin^3 x)\sin x\\ &= \cos^5 x - 6\cos^3 x \sin^2 x + \cos x \sin^4 x\\ &\qquad - 4\cos^3 x \sin^2 x + 4\cos x \sin^4 x\\ &= \cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x. \qquad (5b)\end{aligned}$$

38. RELATION TO NEWTON'S BINOMIAL FORMULA

To simplify the search for regularity in the formulas derived in section 37, we shall write down the results we have obtained in a special table:

$\sin 2x =$	$2 \sin x \cos x$
$\cos 2x =$	$\cos^2 x - \sin^2 x$
$\sin 3x =$	$3 \cos^2 x \sin x - \sin^3 x$
$\cos 3x =$	$\cos^3 x - 3 \cos x \sin^2 x$
$\sin 4x =$	$4 \cos^3 x \sin x - 4 \cos x \sin^3 x$
$\cos 4x =$	$\cos^4 x - 6 \cos^2 x \sin^2 x + \sin^4 x$
$\sin 5x =$	$5 \cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x$
$\cos 5x =$	$\cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^4 x$

In this table we arranged the formulas in pairs, placing the expressions for the sine and cosine of the same multiple of x between two parallel lines. If in each section of the table we read the coefficients of the terms in the order shown by the arrows (ignoring the signs of the coefficients), we recognize them as the well-known binomial coefficients. Thus the sections of the table give the following coefficients:

1st (for $\sin 2x$ and $\cos 2x$): 1, 2, 1;
2nd (for $\sin 3x$ and $\cos 3x$): 1, 3, 3, 1;
3rd (for $\sin 4x$ and $\cos 4x$): 1, 4, 6, 4, 1;
4th (for $\sin 5x$ and $\cos 5x$): 1, 5, 10, 10, 5, 1.

But these are just the numbers in the third, fourth, fifth, and sixth rows of Pascal's triangle (p. 22), that is, the binomial coefficients for $m = 2, 3, 4$ and 5.

It is difficult not to yield to the impression that the expressions for $\cos mx$ and $\sin mx$ are two halves of Newton's binomial formula. Let

us consider, for instance, $\cos 5x$ and $\sin 5x$. If we raise the binomial $\cos x + \sin x$ to the fifth power, we obtain

$$(\cos x + \sin x)^5 = \boxed{\cos^5 x} + 5\cos^4 x \sin x + \boxed{10\cos^3 x \sin^2 x} + 10\cos^2 x \sin^3 x + \boxed{5\cos x \sin^4 x} + \sin^5 x.$$

Comparing this formula with our table, we see clearly that the framed terms (the odd-numbered terms) differ from the terms in the expression for $\cos 5x$ only in their signs. In the same way the even-numbered terms differ only in their signs from the terms of the formula for $\sin 5x$.

We can make analogous observations for $\cos 4x$ and $\sin 4x$, $\cos 3x$ and $\sin 3x$, etc. As a result, we come to the following rule: *If we expand $(\cos x + \sin x)^m$ by the binomial formula, then the odd-numbered terms, written with alternating signs $+ - + - + \ldots$, will give us the expression for $\cos mx$, and the even-numbered terms, also written with alternating signs, will give the expression for $\sin mx$.*

We shall apply this rule to find formulas for $\cos 6x$ and $\sin 6x$:

$$(\cos x + \sin x)^6 = \boxed{\cos^6 x} + 6\cos^5 x \sin x + \boxed{15\cos^4 x \sin^2 x} + 20\cos^3 x \sin^3 x + \boxed{15\cos^2 x \sin^4 x} + 6\cos x \sin^5 x + \boxed{\sin^6 x.}$$

Here the odd-numbered terms are:

$$\cos^6 x, \quad 15\cos^4 x \sin^2 x, \quad 15\cos^2 x \sin^4 x, \quad \sin^6 x.$$

Writing them down with alternating signs we should, according to our rule, obtain an expression for $\cos 6x$:

$$\cos 6x = \cos^6 x - 15\cos^4 x \sin^2 x + 15\cos^2 x \sin^4 x - \sin^6 x.$$

Analogously, we obtain the following formula for $\sin 6x$:

$$\sin 6x = 6\cos^5 x \sin x - 20\cos^3 x \sin^3 x + 6\cos x \sin^5 x.$$

We can verify these formulas using only the formulas for the sine and cosine of the sum of two angles. Note that

$$\sin 6x = \sin(5x + x) = \sin 5x \cos x + \cos 5x \sin x$$

and

$$\cos 6x = \cos(5x + x) = \cos 5x \cos x - \sin 5x \sin x$$

and substitute the expressions previously found for $\cos 5x$ and $\sin 5x$.

The rule which we have found by incomplete induction (verifying its correctness for particular cases) can be written down as a formula. Applying this rule to obtain expressions for cos mx and sin mx for any positive integer m, we first write

$$(\cos x + \sin x)^m = \boxed{\cos^m x} + \frac{m}{1}\cos^{m-1}x \sin x$$
$$+ \boxed{\frac{m(m-1)}{1\cdot 2}\cos^{m-2}\sin^2 x}$$
$$+ \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cos^{m-3}x \sin^3 x$$
$$+ \boxed{\frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}\cos^{m-4}x \sin^4 x}$$
$$+ \frac{m(m-1)(m-2)(m-3)(m-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cos^{m-5}x \sin^5 x$$
$$+ \cdots$$

and then we separate the odd- and even-numbered terms and alternate their signs, obtaining

$$\cos mx = \cos^m x - \frac{m(m-1)}{1\cdot 2}\cos^{m-2}x \sin^2 x$$
$$+ \frac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}\cos^{m-4}x \sin^4 x - \cdots, \qquad (6a)$$

$$\sin mx = \frac{m}{1}\cos^{m-1}x \sin x - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cos^{m-3}x \sin^3 x$$
$$+ \frac{m(m-1)(m-2)(m-3)(m-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cos^{m-5}x \sin^5 x - \cdots. \qquad (6b)$$

These formulas, which include the previously obtained ones as particular cases, were known by Vieta[1] (1590).

[1] François Vieta (1540–1603) was a French mathematician. He was a lawyer by profession, but in spite of a large legal practice, he had time for extensive mathematical studies. His basic work was concerned with algebra and trigonometry, to which he made many contributions. His main historical achievement is his systematic use of letters to denote quantities. Up to his time only unknown quantities had been denoted by letters.

39. POWER SERIES EXPRESSIONS FOR SINE AND COSINE

With the aid of formulas (6*a*) and (6*b*), which are not usually considered in high school, we shall find power series expressions for $\cos x$ and $\sin x$. But for this we still need a number of transformations. First we apply formulas (6*a*) and (6*b*) to $\cos x$ and $\sin x$, by considering x as $m \cdot \frac{x}{m}$. Then

$$\sin x = \sin m\left(\frac{x}{m}\right), \quad \cos x = \cos m\left(\frac{x}{m}\right),$$

and we can substitute $\frac{x}{m}$ for x in formulas (6*a*) and (6*b*). We obtain

$$\cos x = \cos^m \frac{x}{m} - \frac{m(m-1)}{1 \cdot 2} \cos^{m-2} \frac{x}{m} \sin^2 \frac{x}{m} + \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cos^{m-4} \frac{x}{m} \sin^4 \frac{x}{m} - \cdots, \quad (7a)$$

$$\sin x = \frac{m}{1} \cos^{m-1} \frac{x}{m} \sin \frac{x}{m} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \cos^{m-3} \frac{x}{m} \sin^3 \frac{x}{m} + \frac{m(m-1)(m-2)(m-3)(m-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cos^{m-5} \frac{x}{m} \sin^5 \frac{x}{m} - \cdots \quad (7b)$$

These formulas hold for any value of m. Substituting $m = 2, 3, 4, 5$, etc., in formulas (7*a*) and (7*b*), we obtain an infinite number of different expressions for $\cos x$ and $\sin x$. Thus, for $m = 2, 3, 4, 5$ we obtain these expressions for $\cos x$:

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$$

$$= \cos^3 \frac{x}{3} - 3 \cos \frac{x}{3} \sin^2 \frac{x}{3}$$

$$= \cos^4 \frac{x}{4} - 6 \cos^2 \frac{x}{4} \sin^2 \frac{x}{4} + \sin^4 \frac{x}{4}$$

$$= \cos^5 \frac{x}{5} - 10 \cos^3 \frac{x}{5} \sin^2 \frac{x}{5} + 5 \cos \frac{x}{5} \sin^4 \frac{x}{5},$$

and for $\sin x$ the expressions:

$$\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2}$$

$$= 3\cos^2\frac{x}{3}\sin\frac{x}{3} - \sin^3\frac{x}{3}$$

$$= 4\cos^3\frac{x}{4}\sin\frac{x}{4} - 4\cos\frac{x}{4}\sin^3\frac{x}{4}$$

$$= 5\cos^4\frac{x}{5}\sin\frac{x}{5} - 10\cos^2\frac{x}{5}\sin^3\frac{x}{5} + \sin^5\frac{x}{5}.$$

The greater m is, the longer the expressions become. At first glance it may appear strange that, wishing to obtain the most convenient expression for $\cos x$ and $\sin x$, we should increase m without bound in formulas (7*a*) and (7*b*) and give preference to infinitely long formulas (series) over formulas with a finite number of terms. However, we shall soon see that this is convenient.

In accordance with our intentions, we shall continue to increase m in formulas (7*a*) and (7*b*) indefinitely. As a result each term will vary and the number of terms will increase indefinitely.

We shall first examine the changes in the individual terms, taking the series for $\sin x$. The first term is

$$m\cos^{m-1}\frac{x}{m}\sin\frac{x}{m}.$$

As m increases, $\frac{x}{m}$ becomes smaller and smaller, approaching zero. Therefore $\cos\frac{x}{m}$ will approach one and $\sin\frac{x}{m}$ will approach zero (Fig. 18). In the next section we shall show that in spite of this, $m\sin\frac{x}{m}$ approaches x and $\cos^{m-1}\frac{x}{m}$ approaches one. Then it will be evident that the first term of the expansion for $\sin x$,

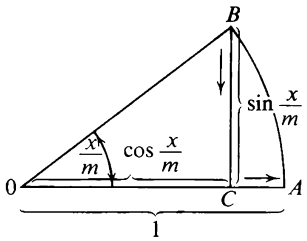


Fig. 18

$$m\cos^{m-1}\frac{x}{m}\sin\frac{x}{m} = \cos^{m-1}\frac{x}{m}\,m\sin\frac{x}{m},$$

approaches $1 \cdot x = x$.

We can write the second term of the expression for $\sin x$, which in formula (7*b*) has the form

$$\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cos^{m-3}\frac{x}{m}\sin^3\frac{x}{m},$$

in the form

$$\frac{m\cdot m\left(1-\frac{1}{m}\right)m\left(1-\frac{2}{m}\right)}{1\cdot 2\cdot 3}\cos^{m-3}\frac{x}{m}\sin^3\frac{x}{m}.$$

$\left(\text{We factored out } m \text{ in the expression } (m-1)\text{: } m-1=m\left(1-\frac{1}{m}\right),\right.$ and also in the expression $(m-2)$: $\left.m-2=m\left(1-\frac{2}{m}\right).\right)$

Now regroup the factors thus:

$$\frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{1\cdot 2\cdot 3}\cos^{m-3}\frac{x}{m}\,m^3\sin^3\frac{x}{m}$$

$$=\frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{1\cdot 2\cdot 3}\cos^{m-3}\frac{x}{m}\left(m\sin\frac{x}{m}\right)^3.$$

In the expression

$$\frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{1\cdot 2\cdot 3}\cos^{m-3}\frac{x}{m}\left(m\sin\frac{x}{m}\right)^3,$$

$1-\frac{1}{m}$ approaches 1 and $1-\frac{2}{m}$ approaches 1, since $\frac{1}{m}$ and $\frac{2}{m}$ approach zero as m increases indefinitely. Also[1]

$$\cos^{m-3}\frac{x}{m}\to 1 \quad \text{and} \quad m\sin\frac{x}{m}\to x,$$

as we shall show in section 40. Therefore we have

$$\frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{1\cdot 2\cdot 3}\cos^{m-3}\frac{x}{m}\left(m\sin\frac{x}{m}\right)^3\to\frac{1\cdot 1}{1\cdot 2\cdot 3}\cdot 1\cdot(x)^3$$

$$=\frac{x^3}{1\cdot 2\cdot 3}.$$

[1] For brevity, we replace the word "approaches" by an arrow "$\to$".

Similar considerations show us that the third term in the expression for $\sin x$,

$$\frac{m(m-1)(m-2)(m-3)(m-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cos^{m-5}\frac{x}{m}\sin^5\frac{x}{m},$$

approaches

$$\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5};$$

the fourth term,

$$\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$
$$\times\cos^{m-7}\frac{x}{m}\sin^7\frac{x}{m},$$

approaches

$$\frac{x^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7};$$

and so on.

Instead of performing the transformations in each case, we shall map out the necessary steps; for instance, we shall show the steps for the fourth term of expression (7*b*). First we rewrite the coefficient of this term,

$$\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7},$$

by factoring m from each of the binomials $m-1$, $m-2$, $m-3$, $m-4$, $m-5$, $m-6$,

$$\frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\left(1-\frac{3}{m}\right)\left(1-\frac{4}{m}\right)\left(1-\frac{5}{m}\right)\left(1-\frac{6}{m}\right)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}.$$

Transposing m^7 to stand beside $\sin^7\frac{x}{m}$, we have

$$m^7\sin^7\frac{x}{m}=\left(m\sin\frac{x}{m}\right)^7.$$

Thus, as a result, the entire fourth term can be expressed in the

following manner:

$$\frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\left(1-\frac{3}{m}\right)\left(1-\frac{4}{m}\right)\left(1-\frac{5}{m}\right)\left(1-\frac{6}{m}\right)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} \times \cos^{m-7}\frac{x}{m}\left(m\sin\frac{x}{m}\right)^7.$$

Finally, we observe that as m increases indefinitely,

$$1-\frac{1}{m}\to 1,\quad 1-\frac{2}{m}\to 1,\quad 1-\frac{3}{m}\to 1,$$
$$1-\frac{4}{m}\to 1,\quad 1-\frac{5}{m}\to 1,\quad 1-\frac{6}{m}\to 1,$$
$$\cos^{m-7}\frac{x}{m}\to 1,\quad m\sin\frac{x}{m}\to x,$$

and therefore the whole expression above approaches

$$\frac{1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\cdot 1\cdot x^7=\frac{x^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}.$$

Thus we see that when m increases indefinitely, the terms of the right-hand part of the formula

$$\begin{aligned}\sin x = {} & \frac{x}{m}\cos^{m-1}\frac{x}{m}\sin\frac{x}{m}\\ & -\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cos^{m-3}\frac{x}{m}\sin^3\frac{x}{m}\\ & +\frac{m(m-1)(m-2)(m-3)(m-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cos^{m-5}\frac{x}{m}\sin^5\frac{x}{m}\\ & -\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\\ & \qquad\times\cos^{m-7}\frac{x}{m}\sin^7\frac{x}{m}\\ & +\cdots \qquad (7b)\end{aligned}$$

become closer and closer to the terms of the power series

$$x-\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{x^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+\cdots.$$

And since the right-hand side of formula (7*b*) for any values of m, no matter how large, is equal to $\sin x$, it is reasonable to suppose that this series also represents $\sin x$, that is,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots . \tag{8a}$$

Similarly, considering one by one the terms of expression for the cosine (7*a*), we may come to the formula

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots . \tag{8b}$$

40. VERIFICATION OF DETAILS IN SECTION 39

We shall prove formulas (8*a*) and (8*b*) later, but first we must verify the correctness of the assertions that $m \sin \frac{x}{m}$ approaches x, and that each of $\cos^m \frac{x}{m}$, $\cos^{m-1} \frac{x}{m}$, $\cos^{m-2} \frac{x}{m}$, $\cos^{m-3} \frac{x}{m}$, ... approaches 1 as m increases indefinitely.

We have already seen that $\sin \frac{x}{m} \to 0$ as $m \to \infty$. We might draw the hasty conclusion that $m \sin \frac{x}{m} \to 0$, since one of the factors tends to zero. But the conclusion would be incorrect. The magnitude of the product $m \cdot \sin \frac{x}{m}$ depends on *both* factors at the same time, and while the second factor tends to zero the first one (m) increases indefinitely. The simplest examples show that just about anything can happen when we make one factor grow rapidly while the other becomes very small. For example, if we take m^2 for one factor and $\frac{1}{m^2}$ for the other and substitute for m larger and larger numbers, then one of the factors will increase indefinitely, while the other one will decrease indefinitely, but their product will remain constant and equal to 1:

$$m^2 \frac{1}{m^2} = 1.$$

Here is another example. Let us assume that one factor is m^2 and the second is $\frac{1}{m^3}$. Then their product

$$m^2 \cdot \frac{1}{m^3} = \frac{m^2}{m^3} = \frac{1}{m}$$

approaches zero as m increases indefinitely.

Finally, let the first factor be m^2 as before and let the second be $\frac{1}{m}$. In this case, the product is

$$m^2 \cdot \frac{1}{m} = \frac{m^2}{m} = m,$$

which increases indefinitely together with m.

Now let us return to our problem and use some geometric facts to help solve it. In Figure 19 we have an angle AOB equal to $\frac{x}{m}$, whose vertex is the center of a circle with radius 1. We construct, at point A, the tangent to the circle so that it intersects the extension of the radius OB. We denote this point of intersection by D. As we know, the segment AD is equal to the tangent of the angle $\frac{x}{m}$; that is,

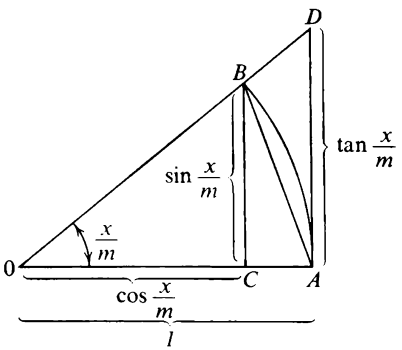


Fig. 19

$$AD = \tan\frac{x}{m} = \frac{\sin\frac{x}{m}}{\cos\frac{x}{m}}.$$

Finally, we draw the chord AB in our figure, and compare the areas of the triangle OAB, the circular sector OAB, and the triangle OAD. Evidently,

$$\text{area of } \triangle OAB < \text{area of sector } OAB < \text{area of } \triangle OAD.$$

But

$$\text{area of } \triangle OAB = \frac{1}{2}\, OA \cdot BC = \frac{1}{2} \cdot 1 \cdot \sin\frac{x}{m} = \frac{1}{2} \cdot \sin\frac{x}{m},$$

$$\text{area of sector } OAB = \frac{1}{2}\, OA \cdot \widehat{BA} = \frac{1}{2} \cdot 1 \cdot \frac{x}{m} = \frac{1}{2} \cdot \frac{x}{m},$$

$$\text{area of } \triangle OAD = \frac{1}{2}\, OA \cdot AD = \frac{1}{2} \cdot 1 \cdot \tan\frac{x}{m} = \frac{1}{2} \cdot \frac{\sin\frac{x}{m}}{\cos\frac{x}{m}}.$$

Therefore,

$$\frac{1}{2}\sin\frac{x}{m} < \frac{1}{2} \cdot \frac{x}{m} < \frac{1}{2}\frac{\sin\frac{x}{m}}{\cos\frac{x}{m}}.$$

(We consider the angle $\frac{x}{m}$ as acute and positive.) The last inequality will remain true if we multiply each of its terms by the positive number $2m$. Then we obtain

$$m \sin\frac{x}{m} < x < \frac{m \sin\frac{x}{m}}{\cos\frac{x}{m}}.$$

(We chose the multiplier $2m$ to obtain a relation containing the expression $m \sin\frac{x}{m}$, which we shall now consider.)

What can we now assert concerning $m \sin\frac{x}{m}$? First (looking at the first two terms of the inequality), that

$$m \sin\frac{x}{m} < x;$$

second (on the basis of the last two terms of the inequality), that

$$x < \frac{m \sin\frac{x}{m}}{\cos\frac{x}{m}}$$

or, multiplying each term by the positive number $\cos\frac{x}{m}$, that

$$x \cos\frac{x}{m} < m \sin\frac{x}{m}.$$

Thus, on the one hand, $m \sin \frac{x}{m}$ for any m is less than x. On the other hand, $m \sin \frac{x}{m}$, also for any m, is greater than $x \cos \frac{x}{m}$. But as m increases indefinitely $\cos \frac{x}{m}$ approaches one and therefore $x \cos \frac{x}{m}$ approaches x (always, however, remaining less than x, since $\cos \frac{x}{m}$ is less than one for all $x \neq 0$). Consequently, $m \sin \frac{x}{m}$, which is less than x and at the same time greater than quantity $x \cos \frac{x}{m}$, which tends to the limit x as m increases, is, so to speak, squeezed up to the limit x itself; and this is what we wanted to establish.

It remains to be shown that $\cos^m \frac{x}{m}$, $\cos^{m-1} \frac{x}{m}$, $\cos^{m-2} \frac{x}{m}$, $\cos^{m-3} \frac{x}{m}$, . . . approach one as m increases indefinitely. At first glance, it appears that the solution here is simpler than in the case we just considered. In fact, once we know that $\cos \frac{x}{m}$ approaches one, then any power of it will also approach one. This conclusion is valid when we are concerned with any definite power—second, third, etc.

In $\cos^m \frac{x}{m}$, however, the exponent itself does not remain constant but increases indefinitely. Now we are again operating on two changing factors, as in the case $m \cdot \sin \frac{x}{m}$. Namely, when m increases indefinitely, then $\cos \frac{x}{m}$ increases, always approaching one (Fig. 18 on p. 119). This taken by itself would show that any single factor of $\cos \frac{x}{m}$ would increase as m increases. But if there should be a longer product of fixed factors with each factor less than one, then the product would decrease in value as the number of factors increased. But we are considering a product with an increasing number of factors, each factor increasing in value yet bounded above by 1:

$$\underbrace{\cos\frac{x}{m}\cdot\cos\frac{x}{m}\cdot\cos\frac{x}{m}\cdots\cos\frac{x}{m}}_{m\text{ times}}$$

We wonder what we can conclude from the increase of $\cos\frac{x}{m}$, which increases the power, and the increase of the exponent, which decreases it (since the base of the power is less than one).

To settle this question we shall use the following computations. First,

$$\begin{aligned}\cos\frac{x}{m} &= \cos\left(2\cdot\frac{x}{2m}\right)\\ &= \cos^2\frac{x}{2m} - \sin^2\frac{x}{2m}\\ &= \left(1 - \sin^2\frac{x}{2m}\right) - \sin^2\frac{x}{2m} = 1 - 2\sin^2\frac{x}{2m};\end{aligned}$$

but

$$\sin\frac{x}{2m} < \frac{x}{2m}.$$

Therefore,

$$\sin^2\frac{x}{2m} < \frac{x^2}{4m^2} \quad\text{and}\quad 2\sin^2\frac{x}{2m} < 2\cdot\frac{x^2}{4m^2} = \frac{x^2}{2m^2}.$$

If in the expression for $\cos\frac{x}{m}$ we replace the subtrahend $2\sin^2\frac{x}{2m}$ by a greater number we decrease the difference; that is,

$$\cos\frac{x}{m} = 1 - 2\sin^2\frac{x}{2m} > 1 - \frac{x^2}{2m^2},$$

and therefore

$$\begin{aligned}\cos^m\frac{x}{m} = \left(1 - 2\sin^2\frac{x}{m}\right)^m &> \left(1 - \frac{x^2}{2m^2}\right)^m\\ &= 1 - \frac{m}{1}\left(\frac{x^2}{2m^2}\right) + \frac{m(m-1)}{1\cdot 2}\left(\frac{x^2}{2m^2}\right)^2\\ &\quad - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\left(\frac{x^2}{2m^2}\right)^3 + \cdots\end{aligned}$$

(by the binomial formula). We can only decrease the right-hand side

of this inequality if we change all + signs (except the sign of the first term) to − signs. Consequently,

$$\cos^m \frac{x}{m} > 1 - \left(\frac{m}{1}\right)\left(\frac{x^2}{2m^2}\right) - \frac{m(m-1)}{1\cdot 2}\left(\frac{x^2}{2m^2}\right)^2 - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\left(\frac{x^2}{2m^2}\right)^3 - \cdots.$$

Further, in each of the fractions which are subtracted from one, we shall increase the numerator and decrease the denominator. Namely, everywhere in the numerators we replace $m-1$, $m-2$, $m-3$, ... by the larger number m, and in the denominators replace 2, 3, 4, ... by the smaller number 1. This increases every subtracted fraction and the whole right-hand side of the inequality will be changed into a still smaller number. We obtain the stronger inequality

$$\begin{aligned}\cos^m \frac{x}{m} &> 1 - \frac{m}{1}\left(\frac{x^2}{2m^2}\right) - \frac{m\cdot m}{1\cdot 1}\left(\frac{x^2}{2m^2}\right)^2 - \frac{m\cdot m\cdot m}{1\cdot 1\cdot 1}\left(\frac{x^2}{2m^2}\right)^3 - \cdots \\ &= 1 - m\frac{x^2}{2m^2} - m^2\frac{x^4}{4m^4} - m^3\frac{x^6}{8m^6} - \cdots \\ &= 1 - \frac{x^2}{2m} - \frac{x^4}{4m^2} - \frac{x^6}{8m^3} - \cdots \\ &= 1 - \frac{x^2}{2m}\left(1 + \frac{x^2}{2m} + \frac{x^4}{4m^2} + \cdots\right).\end{aligned}$$

The terms in parentheses form a geometric progression with the common ratio $\frac{x^2}{2m}$. If this ratio is less than one (which is sure to occur when m is a sufficiently large number) and if we consider the progression extended indefinitely (which only makes the inequality stronger), then, using the formula for the sum of an infinite geometric progression, we obtain

$$\begin{aligned}\cos^m \frac{x}{m} &> 1 - \frac{x^2}{2m}\cdot\frac{1}{1-\frac{x^2}{2m}} \\ &= 1 - \frac{x^2}{2m - x^2}.\end{aligned}$$

If now m increases indefinitely, then the denominator of the fraction

$$\frac{x^2}{2m - x^2}$$

also increases and the whole fraction approaches zero. Consequently, the quantity

$$1 - \frac{x^2}{2m - x^2}$$

approaches one and the quantity $\cos^m \frac{x}{m}$, which on the one hand is less than one $\left(\text{since } \cos \frac{x}{m} < 1\right)$ and on the other hand is greater than the number $1 - \frac{x^2}{2m - x^2}$, which tends to the limit 1, must itself tend to the limit 1; that is, $\cos^m \frac{x}{m} \to 1$, when $m \to \infty$. But now we also find that

$$\cos^{m-1} \frac{x}{m}, \quad \cos^{m-2} \frac{x}{m}, \quad \cos^{m-3} \frac{x}{m},$$

tend to one. Indeed, we have the following inequalities:

$$\cos^m \frac{x}{m} < \cos^{m-1} \frac{x}{m} < 1,$$

$$\cos^m \frac{x}{m} < \cos^{m-2} \frac{x}{m} < 1,$$

$$\cos^m \frac{x}{m} < \cos^{m-3} \frac{x}{m} < 1,$$

since the greater the exponent of the number $\cos \frac{x}{m} < 1$, the smaller the result. Therefore

$$\cos^{m-1} \frac{x}{m}, \quad \cos^{m-2} \frac{x}{m}, \quad \cos^{m-3} \frac{x}{m}, \quad \ldots,$$

contained between one and $\cos^m \frac{x}{m}$ (which, as we have shown, approaches one), must themselves approach one.

41. PROOF OF SERIES EXPANSION FOR sin x AND cos x

We have established that if we increase m indefinitely in the right-hand sides of the formulas

$$\cos x = \cos^m \frac{x}{m} - \frac{m(m-1)}{1 \cdot 2} \cos^{m-2} \frac{x}{m} \cdot \sin^2 \frac{x}{m} + \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cos^{m-4} \frac{x}{m} \sin^4 \frac{x}{m} - \cdots, \quad (7a)$$

$$\sin x = \frac{m}{1} \cos^{m-1} \frac{x}{m} \sin \frac{x}{m} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \cos^{m-3} \frac{x}{m} \sin^3 \frac{x}{m} + \cdots, \quad (7b)$$

then, first, the number of terms in the right-hand part will increase indefinitely, and second, each of the terms will tend to the corresponding term of the power series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots, \quad (*)$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots. \quad (**)$$

The question is, can we assert that the first power series represents cos x and the second sin x? It turns out that they do, but this problem requires more study. What basis do we have to expect the answer we want? First, the right-hand sides of formulas (7*a*) and (7*b*) for any m, no matter how large, give cos x and sin x, and second, for large m, each of the terms of these right-hand sides differs very little from the terms of the series (*) and (**).

The difficulty that may arise here is once again due to the presence of two factors acting in opposite directions: on the one hand the differences between terms of the series (7*a*) and (7*b*) and of the series (*) and (**) decrease indefinitely, but on the other hand the number of terms increases indefinitely. There are cases, superficially similar to our present situation, in which the increase in the number of terms more than makes up for their increasing individual closeness to their respective limit terms. For the simplest example, let us consider the sum of m equal terms of the form $\frac{1}{m}$:

$$\underbrace{\frac{1}{m} + \frac{1}{m} + \frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}}_{m \text{ times}}.$$

It is evident that for any arbitrarily large m this sum will remain the same, equal to one;

$$\underbrace{\frac{1}{m}+\frac{1}{m}+\frac{1}{m}+\frac{1}{m}+\cdots+\frac{1}{m}}_{m \text{ times}}=1.$$

Now we begin to increase m indefinitely; the number of terms will increase indefinitely and each term will approach zero. If we write the series

$$0+0+0+0+0+0+\cdots,$$

then we see that the terms of our sum approach the terms of this series. But the value of each of our sums is one while the sum of the series is zero.

Thus the considerations we gave above are not in themselves sufficient ground for us to assert that the sums of series (*) and (**) will be the same as the sums of the right-hand sides of formulas (7*a*) and (7*b*), that is, cos x and sin x. However, we can still verify that this is the case.

Suppose that we want to prove that

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots. \qquad (8b)$$

First, we observe that the series on the right-hand side converges, in fact converges absolutely, for any value of x. To prove this, we use d'Alembert's test. We consider the absolute value of the ratio of each term to its predecessor. The absolute value of the ratio of the

second term to the first is $\left|\dfrac{-\frac{x^2}{2!}}{1}\right| = \dfrac{x^2}{1\cdot 2}$,

of the third to the second is $\left|\dfrac{\frac{x^4}{4!}}{-\frac{x^2}{2!}}\right| = \dfrac{x^2}{3\cdot 4}$,

of the fourth to the third is $\left|\dfrac{-\frac{x^6}{6!}}{\frac{x^4}{4!}}\right| = \dfrac{x^2}{5\cdot 6}$,

etc. In general, the ratio of each term to the preceding one has the form

$$\frac{x^2}{(n-1)n}$$

where $n-1$ and n are integers. As we move away from the beginning of the series, the numbers $n-1$ and n increase indefinitely and the number $\frac{x^2}{(n-1)n} \to 0$. Therefore, starting with some term, this ratio will become and remain smaller than any given positive number q less than one. And from this, by d'Alembert's test, follows the absolute convergence of the series.

To prove that the sum of the series is equal to $\cos x$ for any x, we must verify that the partial sums of the series approach $\cos x$ as a limit; that is, that the difference between $\cos x$ and these partial sums can be made less than any arbitrarily small positive number ε by using a sufficiently large number of terms.

We shall make the number of terms in the sum

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \pm \frac{x^n}{n!}$$

$\left(\text{this number is } \frac{n}{2} + 1 = \frac{n+2}{2}\right)$ sufficiently large to make the sum

$$\frac{|x|^{n+2}}{(n+2)!} + \frac{|x|^{n+4}}{(n+4)!} + \frac{|x|^{n+6}}{(n+6)!} + \cdots$$

of the absolute values of the remaining terms of the series less than $\frac{\varepsilon}{2}$. (This can be done because the series converges absolutely.)

We now take in the right-hand side of formula (7*a*) a number m greater than n and we consider the difference

$$\cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \pm \frac{x^n}{n!}\right).$$

If we substitute for $\cos x$ its expression by formula (7*a*) and group the terms by pairs, the first term of the expression for $\cos x$ with the first term of the series, the second with the second, the third with the third, etc., we obtain

$$\left[\cos^m \frac{x}{m} - 1\right] - \left[\frac{m(m-1)}{2!} \cos^{m-2} \frac{x}{m} \sin^2 \frac{x}{m} - \frac{x^2}{2!}\right]$$
$$+ \left[\frac{m(m-1)(m-2)(m-3)}{4!} \cos^{m-4} \frac{x}{m} \sin^4 \frac{x}{m} - \frac{x^4}{4!}\right] \pm \cdots$$
$$\pm \left[\frac{m(m-1)\cdots(m-n+1)}{n!} \cos^{m-n} \frac{x}{m} \sin^n \frac{x}{m} - \frac{x^n}{n!}\right]$$

$$\mp\left[\frac{m(m-1)\cdots(m-n+1)(m-n)(m-n-1)}{(n+2)!}\right.$$

$$\times \cos^{m-n-2}\frac{x}{m}\sin^{n+2}\frac{x}{m}$$

$$-\frac{m(m-1)\cdots(m-n-3)}{(n+4)!}\cos^{m-n-4}\frac{x}{m}\sin^{n+4}\frac{x}{m}$$

$$+\frac{m(m-1)\cdots(m-n-5)}{(n+6)!}\cos^{m-n-6}\frac{x}{m}\sin^{n+6}\frac{x}{m}$$

$$\left.-\cdots\right].$$

The last brackets contain all the terms of expression (7*a*) for $\cos x$ which had no corresponding terms in the partial sum

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\pm\frac{x^n}{n!}$$

with which to form pairs. (Remember that we have taken $m > n$. Therefore the right-hand side of formula (7*a*) contains more terms than our partial sum.)

Every pair of brackets except the last one contains the difference between two terms, one from formula (7*a*), the other from series (*). But the first terms, as we have seen, approach the others as limits when m increases indefinitely. We may therefore choose m so large that each of these bracketed terms has absolute value less than $\frac{\varepsilon}{n+2}$. Then together they give us a sum whose absolute value is less than $\frac{n+2}{2}\cdot\frac{\varepsilon}{n+2}=\frac{\varepsilon}{2}$.

In the last pair of brackets, which contains all the terms of expression (7*a*) without mates in the partial sum, the number of terms may be very large (if we were forced to choose m very large). However, we shall now show that the absolute value of each of them will be less than that of the corresponding term of the series

$$\frac{|x|^{n+2}}{(n+2)!}+\frac{|x|^{n+4}}{(n+4)!}+\frac{|x|^{n+6}}{(n+6)!}+\cdots$$

whose sum is less than $\frac{\varepsilon}{2}$ (p. 132).

For example, the first of these terms,

$$\frac{m(m-1)(m-2)\cdots(m-n-1)}{(n+2)!}\cos^{m-n-2}\frac{x}{m}\sin^{n+2}\frac{x}{m},$$

can be written in the form

$$\frac{m\cdot m\left(1-\frac{1}{m}\right)m\left(1-\frac{2}{m}\right)\cdots m\left(1-\frac{n+1}{m}\right)}{(n+2)!}$$

$$\times\cos^{m-n-2}\frac{x}{m}\sin^{n+2}\frac{x}{m}$$

$$=\frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{n+1}{m}\right)}{(n+2)!}$$

$$\times\cos^{m-n-2}\frac{x}{m}\cdot m^{n+2}\cdot\sin^{n+2}\frac{x}{m}.$$

But each of the numbers

$$1-\frac{1}{m},\quad 1-\frac{2}{m},\quad\cdots,\quad 1-\frac{n+1}{m}$$

is smaller than one. Also,

$$\left|\cos\frac{x}{m}\right|<1$$

and (as we have seen on p. 125)

$$\left|m\sin\frac{x}{m}\right|<|x|.$$

Therefore, the absolute value of the expression under consideration is less than

$$\frac{1\cdot 1\cdot\ldots\cdot 1}{(n+2)!}\cdot 1^{m-n-2}\cdot|x|^{n+2}=\frac{|x|^{n+2}}{(n+2)!}.$$

Analogously, we find that the absolute value of the second term in the last pair of brackets (pp. 132–133) is less than

$$\frac{|x|^{n+4}}{(n+4)!},$$

of the third, it is less than

$$\frac{|x|^{n+6}}{(n+6)!},$$

and so on. But this means that the absolute value of the whole expression in brackets is less than the sum of the series

$$\frac{|x|^{n+2}}{(n+2)!} + \frac{|x|^{n+4}}{(n+4)!} + \frac{|x|^{n+6}}{(n+6)!} + \cdots,$$

which is less than $\frac{\varepsilon}{2}$.

Considering this result, and taking into account that the sum of the absolute values of the remaining brackets is also less than $\frac{\varepsilon}{2}$, we conclude that the value of the whole expression on p. 132,

$$\cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \pm \frac{x^n}{n!}\right),$$

is less than ε $\left(\text{for } n \text{ selected so large that } \frac{|x|^{n+2}}{(n+2)!} + \frac{|x|^{n+4}}{(n+4)!} + \frac{|x|^{n+6}}{(n+6)!} + \cdots < \frac{\varepsilon}{2}\right)$.

And since ε is an arbitrarily small positive number it follows from this that

$$\lim \left[\cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \pm \frac{x^n}{n!}\right)\right] = 0,$$

or that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \pm \frac{x^n}{n!} \mp \frac{x^{n+2}}{(n+2)!} \pm \frac{x^{n+4}}{(n+4)!} \mp \cdots,$$

which is what we wanted to prove.

In the same way we can establish that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots.$$

It is first necessary to prove that the series in the right-hand side is absolutely convergent for any x (by d'Alembert's test), and then that the sum of this series equals $\sin x$; that is, that the absolute value of the difference between $\sin x$ and the partial sum

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \pm \frac{x^n}{n!}$$

can be made smaller than any arbitrarily small number ε by taking n sufficiently large.

We shall show a few of the more important applications of the series derived above.

42. COMPUTATION OF VALUES OF THE SINE AND COSINE

Series (8*a*) and (8*b*) make it possible to compute values of the cosine and sine of a given angle. We must remember that in our formulas the angle x is measured in radians. Therefore, if an angle is given in degree measure, we must first change it into radian measure and then use the series. Observe that in computations we can always limit ourselves to x positive and not greater than $\frac{\pi}{4}(45°)$, because trigonometric functions of arbitrary arguments can always be reduced to functions of positive arguments not greater than $\frac{\pi}{4} = 0.785398\ldots(45°)$. For example,

$$\cos(-0.3) = \cos 0.3,$$

$$\sin(-15°) = -\sin 15°,$$

$$\begin{aligned}\cos 10.27 &= \cos(10.27 - 6.28) \quad (\text{where } 6.28 \approx 2\pi)\\ &= \cos 3.99\\ &= \cos(4.71 - 0.72) \quad (\text{where } 4.71 \approx \tfrac{3}{2}\pi)\\ &= -\sin 0.72, \text{ etc.}\end{aligned}$$

But if $0 \leq x \leq \frac{\pi}{4}$, then both series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

will be alternating with decreasing terms. (Indeed, the denominators increase and the numerators decrease, being powers with increasing exponents of a number x which is less than one ($x < 0.785398\ldots$).) Consequently, by discarding from each of the series all terms beginning with a certain one we make an error smaller than the first of the discarded terms. Therefore, if we need to compute the values of $\cos x$ and $\sin x$ accurate to within 0.0005,[1] we can, even in the most unfavorable case of the largest x ($x = \frac{\pi}{4} = 0.785398\ldots$), limit our-

[1] That is, to one half of the third digit after the decimal point; three-digit trigonometric tables give the values of functions with this accuracy.

selves to the first three terms of both series:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}, \tag{9a}$$

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}. \tag{9b}$$

In fact, in the first case, the absolute value of the first discarded term equals $\frac{x^7}{7!} = \frac{x^7}{5040}$. For $x = \frac{\pi}{4} = 0.785\ldots$,

$$\frac{x^7}{5040} = \frac{(0.785\ldots)^7}{5040} < \frac{(0.79)^7}{5040} = \frac{0.192039\ldots}{5040} < \frac{0.2}{5000} = 0.00004.$$

In the second case (for $\cos x$), the absolute value of the first rejected term equals $\frac{x^6}{6!} = \frac{x^6}{720}$. For $x = \frac{\pi}{4} = 0.785\ldots$,

$$\frac{x^6}{720} = \frac{(0.785\ldots)^6}{720} < \frac{(0.79)^6}{720} < \frac{0.243087\ldots}{720} < 0.00035.$$

We need to take this number of terms, however, only if x is close to $\frac{\pi}{4} = 0.785398\ldots$. For smaller arguments the same accuracy may be secured with a smaller number of terms.

Let us find out for what arguments we can take only the first two terms from the series for $\cos x$ and $\sin x$, that is, when we can use the approximate formulas

$$\sin x \approx x - \frac{x^3}{6}, \tag{10a}$$

$$\cos x \approx 1 - \frac{x^2}{2}, \tag{10b}$$

assuming we want to retain the same accuracy (to 0.0005). Since the error in the first formula is less than $\frac{x^5}{5!}$, we need to find the values of x for which

$$\frac{x^5}{5!} < 0.0005.$$

Since $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$ and $x^5 < 0.0005 \cdot 120 = 0.06$,

this means that

$$x < \sqrt[5]{0.06} = \frac{\sqrt[5]{6000}}{10} = \frac{5.696\ldots}{10} = 0.5696\ldots.$$

Thus for $x < 0.5696\ldots$, which corresponds to angles smaller than $32°38'$, we can use formula (10a) to compute $\sin x$ accurate to 0.0005.

Analogously, we shall find that for $x < 0.3309\ldots$, which corresponds to angles smaller than 18°57′, we can use formula (10*b*) to compute cos x accurate to 0.0005.

Finally, we can use

$$\sin x \approx x, \tag{11a}$$

$$\cos x \approx 1, \tag{11b}$$

accurate up to 0.0005 if for sin x

$$\frac{x^3}{6} < 0.0005$$

or

$$x < 0.1144,$$

which in degree measure corresponds to angles smaller than 6°33′. For cos x, in order to attain the desired accuracy, we should have

$$\frac{x^2}{2} < 0.0005$$

or

$$x < 0.0316\ldots,$$

which in degree measure corresponds to angles smaller than 1°48′.

Thus to compute sin x accurate to 0.0005

$$\text{for} \quad \begin{cases} 0.569 < x < 0.785 \\ 32° < x < 45° \end{cases} \quad \text{we use} \quad \sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}; \tag{9a}$$

$$\text{for} \quad \begin{cases} 0.114 < x < 0.569 \\ 6° < x < 32° \end{cases} \quad \text{we use} \quad \sin x \approx x = \frac{x^3}{6}; \tag{10a}$$

$$\text{for} \quad \begin{cases} 0 \leq x \leq 0.114 \\ 0° \leq x \leq 6° \end{cases} \quad \text{we use} \quad \sin x \approx x. \tag{11a}$$

Analogously, to compute cos x accurate to 0.0005

$$\text{for} \quad \begin{cases} 0.330 < x \leq 0.785 \\ 19° < x \leq 45° \end{cases} \quad \text{we use} \quad \cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}; \tag{9b}$$

$$\text{for} \quad \begin{cases} 0.031 < x < 0.330 \\ 2° < x < 19° \end{cases} \quad \text{we use} \quad \cos x \approx 1 - \frac{x^2}{2}; \tag{10b}$$

$$\text{for} \quad \begin{cases} 0 < x \leq 0.031 \\ 0° < x \leq 2° \end{cases} \quad \text{we use} \quad \cos x \approx 1. \tag{11b}$$

For example,

$$\sin 1^\circ = \sin 0.0175 \approx 0.0175.$$

The error in the formula

$$\sin x = x$$

for $x = 0.0175$ (1°) is less than

$$\frac{(0.0175)^3}{6} = \frac{0.0000053\ldots}{6} < 0.000001;$$

therefore, we should change 1° into radian measure with a greater accuracy than we have done. Observing that

$$1^\circ = 0.0175453 \text{ radians},$$

we find

$$\sin 1^\circ = \sin 0.0175453 \approx 0.017545$$

(the error here is no greater than 0.000001).

Another example:

$$\cos 40^\circ = \cos\left(40 \cdot \frac{\pi}{180}\right) = \cos\frac{2}{9}\pi = \cos 0.6981.$$

By formula (9*b*), we have

$$\cos 40^\circ = \cos 0.6981 \approx 1 - \frac{(0.6981)^2}{2} + \frac{(0.6981)^4}{24}$$
$$\approx 1 - 0.2437 + 0.0099 = 0.7662.$$

This result is certainly accurate to within 0.0005. Actually, in this case the error in the formula is less than

$$\frac{(0.6981)^6}{720} < \frac{(0.7)^6}{720} = \frac{0.117\ldots}{720} < 0.0002.$$

Since we computed the second and the third terms, $-\frac{x^2}{2}$ and $\frac{x^4}{4!}$, accurate to 0.00005 each, the error in the result cannot exceed 0.0003.

Thus

$$\cos 40^\circ = 0.7662 \pm 0.0003.$$

43. SNELL'S METHOD FOR RECTIFYING AN ARC OF A CIRCLE

With the aid of the series for $\sin x$ and $\cos x$ and the previously obtained binomial series, we can derive additional formulas, both accurate and approximate. We shall now give examples of applications of infinite series to some problems of approximate geometric constructions.

Let us consider the problem of rectifying an arc of a circle, that is, constructing with a ruler and compass a segment of a straight line whose length equals the length of a given arc of a circle. It has been proved that this problem can be solved only approximately. There are simple methods (Snell, Rankine), however, which give a solution sufficiently accurate for all practical requirements.

EXAMPLE. Construct a line segment equal in length to arc AB of the circle in Figure 20.

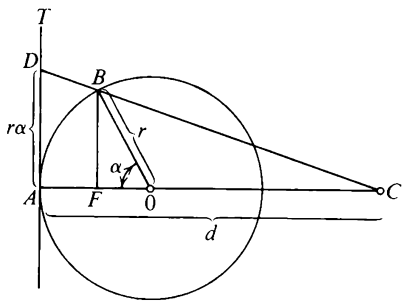


Fig. 20

Solution. Snell[1] solved this problem in the following way. We construct the tangent AT to the circle at the point A on the circle and assume that the segment AD gives a solution; that is, length $AD =$ arc AB. We connect points D and B by a segment DB and extend it until it intersects the extension of the radius AO at a point C. To find the segment AC, we need only know the segment AD. Also vice versa: knowing the segment AC, we can find AD by producing CB to meet the tangent A in D. Thus we shall try to construct the segment AC; for this purpose we need only find its length d. From B we drop

[1] Willebrord Snell (Snellius) (1581–1626) was a professor in Leyden, Holland. He is known as the discoverer of the law of refraction of rays of light.

the perpendicular BF to AO and, because triangles ADC and FBC are similar, we have

$$\frac{AC}{FC} = \frac{AD}{FB}.$$

But

$$AC = d, \qquad AD = \widehat{AB} = r\alpha,$$
$$FC = OC + OF = (d - r) + r\cos\alpha, \qquad FB = r\sin\alpha.$$

Therefore, we have

$$\frac{d}{d - r + r\cos\alpha} = \frac{r\alpha}{r\sin\alpha} = \frac{\alpha}{\sin\alpha},$$

from which we have

$$d\sin\alpha = d\alpha - r\alpha + r\alpha\cos\alpha.$$

Consequently,

$$d\alpha - d\sin\alpha = r\alpha - r\alpha\cos\alpha$$

and

$$d = \frac{r\alpha(1 - \cos\alpha)}{\alpha - \sin\alpha}.$$

Snell's method succeeded because he found a simple but sufficiently good approximation for d. It appears that

$$d \approx 3r.$$

We shall verify this estimate for d and evaluate the error in our estimate of the length of AD caused by the use of this approximation. To do this, we shall use the infinite series expansions of $\sin\alpha$ and $\cos\alpha$:

$$1 - \cos\alpha = 1 - \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \frac{\alpha^8}{8!} - \cdots\right)$$
$$= \frac{\alpha^2}{2!} - \frac{\alpha^4}{4!} + \frac{\alpha^6}{6!} - \frac{\alpha^8}{8!} + \cdots,$$
$$\alpha - \sin\alpha = \alpha - \left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \frac{\alpha^9}{9!} - \cdots\right)$$
$$= \frac{\alpha^3}{3!} - \frac{\alpha^5}{5!} + \frac{\alpha^7}{7!} - \frac{\alpha^9}{9!} + \cdots.$$

Therefore,

$$d = \frac{r\alpha\left(\frac{\alpha^2}{2!} - \frac{\alpha^4}{4!} + \frac{\alpha^6}{6!} - \cdots\right)}{\left(\frac{\alpha^3}{3!} - \frac{\alpha^5}{5!} + \frac{\alpha^7}{7!} - \cdots\right)};$$

factoring, we have

$$d = \frac{r\frac{\alpha^3}{2}\left(1 - \frac{\alpha^2}{3\cdot 4} + \frac{\alpha^4}{3\cdot 4\cdot 5\cdot 6} - \cdots\right)}{\frac{\alpha^3}{6}\left(1 - \frac{\alpha^2}{4\cdot 5} + \frac{\alpha^4}{4\cdot 5\cdot 6\cdot 7} - \cdots\right)}$$

$$= 3r\frac{1 - \frac{\alpha^2}{12} + \frac{\alpha^4}{360} - \cdots}{1 - \frac{\alpha^2}{20} + \frac{\alpha^4}{840} - \cdots}.$$

Dividing the numerator by the denominator according to the rules for dividing a polynomial by another polynomial (all operations with absolutely convergent series can be carried out by the same rules as operations with polynomials), we obtain

$$\begin{array}{r|l} & 1 - \frac{\alpha^2}{30} - \frac{\alpha^4}{12600} - \cdots \\ \hline 1 - \frac{\alpha^2}{20} + \frac{\alpha^4}{840} - \cdots & 1 - \frac{\alpha^2}{12} + \frac{\alpha^4}{360} - \cdots \\ & 1 - \frac{\alpha^2}{20} + \frac{\alpha^4}{840} - \cdots \\ \hline & -\frac{\alpha^2}{30} + \frac{\alpha^4}{630} - \cdots \\ & -\frac{\alpha^2}{30} + \frac{\alpha^4}{600} - \cdots \\ \hline & -\frac{\alpha^4}{12600} + \cdots \end{array}$$

Thus,

$$d = 3r\left(1 - \frac{\alpha^2}{30} - \frac{\alpha^4}{12600} - \cdots\right)$$

$$= r\left(3 - \frac{\alpha^2}{10} - \frac{\alpha^4}{4200} - \cdots\right).$$

We obtain Snell's approximation,

$$d \approx 3r$$

by discarding from this series all the terms except the first one.

Since the series is not alternating, we cannot assert that the error will be smaller than the first discarded term, $\frac{\alpha^2}{10}$. However, from the equation

$$d = 3r \frac{1 - \frac{\alpha^2}{12} + \frac{\alpha^4}{360} - \cdots}{1 - \frac{\alpha^2}{20} + \frac{\alpha^4}{840} - \cdots}$$

it can be shown that

$$3 - \frac{2\alpha^2}{19} < \frac{d}{r} < 3$$

for

$$0 \leq \alpha \leq 1 \ (57°17' \ldots).$$

Thus, for

$$\alpha = 0.7854 \ldots (45°),$$

we would obtain

$$3 - \frac{2\,(0.7854)^2}{19} < \frac{d}{r} < 3$$

or

$$2.935r < d < 3r.$$

Indeed, for $\alpha = \frac{\pi}{4}$, we have

$$d = \frac{r \cdot \frac{\pi}{4}\left(1 - \cos\frac{\pi}{4}\right)}{\frac{\pi}{4} - \sin\frac{\pi}{4}}$$

$$= \frac{r \cdot 0.7854\,(1 - 0.7071)}{0.7854 - 0.7071}$$

$$\approx 2.938r.$$

For smaller angles α, the difference between d and $3r$ is considerably less. But we are interested not in the error in the construction of segment AC but in the error in the construction of AD. This is easy to

evaluate. In fact, if the length of segment AC_1 is $3r$ (Fig. 21), then

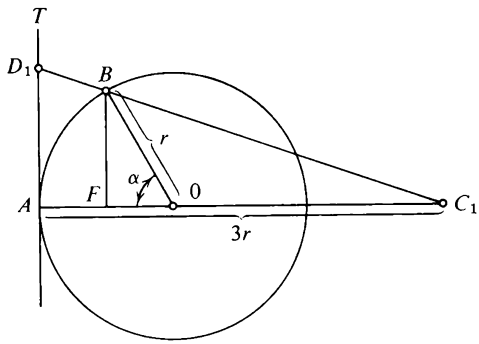


Fig. 21

the corresponding segment AD_1 can be found from the equation

$$\frac{AD_1}{FB} = \frac{AC_1}{FC_1},$$

where

$$FB = r \sin \alpha, \qquad AC_1 = 3r$$

and

$$FC_1 = OC_1 + OF = 2r + r \cos \alpha.$$

Thus

$$\frac{AD_1}{r \sin \alpha} = \frac{3r}{2r + r \cos \alpha} = \frac{3}{2 + \cos \alpha}.$$

If we substitute here for $\sin \alpha$ and $\cos \alpha$ their expansions in infinite series, we find

$$\begin{aligned} AD_1 &= \frac{3r\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots\right)}{2 + \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\right)} \\ &= \frac{3r\alpha\left(1 - \frac{\alpha^2}{3!} + \frac{\alpha^4}{5!} - \cdots\right)}{3\left(1 - \frac{\alpha^2}{2!3} + \frac{\alpha^4}{4!3} - \cdots\right)} \\ &= r\alpha \frac{1 - \frac{\alpha^2}{3!} + \frac{\alpha^4}{5!} - \cdots}{1 - \frac{\alpha^2}{2!3} + \frac{\alpha^4}{4!3} - \cdots} \end{aligned}$$

Thus, we can write

$$AD_1 = r\alpha \frac{1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120} - \cdots}{1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{72} - \cdots}.$$

Dividing the numerator by the denominator, we obtain

$$\begin{array}{r|l}
1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{72} - \frac{\alpha^6}{2160} + \cdots & 1 - \frac{\alpha^4}{180} - \frac{\alpha^6}{1512} - \cdots \\
\hline
 & 1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120} - \frac{\alpha^6}{5040} + \cdots \\
 & \underline{1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{72} - \frac{\alpha^6}{2160}} \\
 & \quad - \frac{\alpha^4}{180} + \frac{\alpha^6}{3780} - \cdots \\
 & \quad \underline{- \frac{\alpha^4}{180} + \frac{\alpha^6}{1080} - \cdots} \\
 & \quad\quad - \frac{\alpha^6}{1512} + \cdots
\end{array}$$

Thus

$$AD_1 = r\alpha\left(1 - \frac{\alpha^4}{180} - \frac{\alpha^6}{1512} - \cdots\right),$$

while we were to construct the segment

$$AD = r\alpha.$$

The difference between the segment we wanted and the one we obtained is the quantity

$$\begin{aligned} AD - AD_1 &= r\alpha - r\alpha\left(1 - \frac{\alpha^4}{180} - \frac{\alpha^6}{1512} - \cdots\right) \\ &= r\frac{\alpha^5}{180} + r\frac{\alpha^7}{1512} + \cdots. \end{aligned}$$

To obtain an estimate for this quantity (the error in the construction), we shall look again at the division of one series by another which we have just carried out, and we shall stop with the first remainder. The quotient will be 1 and the remainder

$$-\frac{\alpha^4}{180} + \frac{\alpha^6}{3780} - \cdots;$$

therefore the expression for AD_1 can be written in the form

$$AD_1 = r\alpha \frac{1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{120} - \frac{\alpha^8}{5040} + \cdots}{1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{72} - \frac{\alpha^6}{2160} + \cdots}$$

$$= r\alpha\left(1 - \frac{\frac{\alpha^4}{180} - \frac{\alpha^6}{3780} + \cdots}{1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{72} - \frac{\alpha^6}{2160} + \cdots}\right).$$

In both the numerator and the denominator of the fraction in parentheses, we have alternating series. The absolute values of the terms of both these series decrease if $\alpha < 1$ (which corresponds to $57°17'\ldots$). Therefore, we shall increase this fraction and at the same time decrease the difference between the terms in the parentheses, if for the numerator we substitute a greater quantity, $\frac{\alpha^4}{180}$, and for the denominator a smaller one, $1 - \frac{\alpha^2}{6}$. We obtain

$$AD_1 > r\alpha\left(1 - \frac{\frac{\alpha^4}{180}}{1 - \frac{\alpha^2}{6}}\right).$$

If $\alpha \leq 1$, then

$$1 - \frac{\alpha^2}{6} \geq \frac{5}{6},$$

$$\frac{\frac{\alpha^4}{180}}{1 - \frac{\alpha^2}{6}} \leq \frac{\frac{\alpha^4}{180}}{\frac{5}{6}}$$

$$= \frac{\alpha^4}{150},$$

and

$$AD_1 > r\alpha\left(1 - \frac{\alpha^4}{150}\right).$$

On the other hand,

$$AD_1 = r\alpha\left(1 - \frac{\frac{\alpha^4}{180} - \frac{\alpha^6}{3780} + \cdots}{1 - \frac{\alpha^2}{6} + \frac{\alpha^4}{72} - \frac{\alpha^6}{2160} + \cdots}\right) < r\alpha.$$

Therefore,

$$r\alpha > AD_1 > r\alpha\left(1 - \frac{\alpha^4}{150}\right)$$
$$= r\alpha - \frac{r\alpha^5}{150}$$

and

$$0 > AD_1 - r\alpha > -\frac{r\alpha^5}{150}.$$

Thus we have established that $AD_1 < r\alpha$ ($r\alpha$ is the length of the arc AB), but the absolute value of the difference $AD_1 - r\alpha$ is not greater than $\frac{r\alpha^5}{150}$.

The relative error

$$\left|\frac{AD_1 - r\alpha}{r\alpha}\right|$$

is less than

$$\frac{r\alpha^5}{150 \cdot r\alpha} = \frac{\alpha^4}{150}.$$

So for $\alpha = 0.7854\ldots$ ($45°$) the relative error does not exceed the quantity

$$\frac{(0.7854)^4}{150} < \frac{0.4}{150} = 0.0026\ldots < 0.003.$$

For smaller angles it is considerably less. We observe that the relative error, being proportional to α^4, decreases $2^4 = 16$ times when the arc becomes twice as small. Therefore, before we apply this method to an arc that is not very small, we should divide it into 2, 4, or 8 parts and construct a segment whose length is very nearly equal to one of these parts of the arc. Then we take it 2, 4, or 8 times this length to obtain the final solution of the problem.

44. RANKINE'S METHOD FOR RECTIFYING AN ARC OF A CIRCLE

To give an example in which we use both the binomial series and the series for sine and cosine, we shall consider Rankine's method for approximately rectifying an arc of a circle.

Solution. We produce the chord BA (Fig. 22) of the given arc by half its length to the point K. Then we draw an arc of the circle with

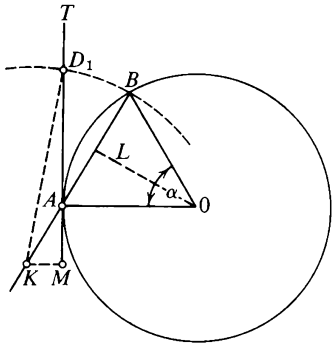


Fig. 22

center K and radius KB to meet the tangent AT to the given circle in the point D_1. The segment AD_1 will be approximately equal to the length of the arc AB.

To evaluate the accuracy of this construction, we shall express the length of AD_1 in terms of r and α and compare the expression we obtain with the length $r\alpha$ of the arc AB.

From the triangle AKD_1, where we drop the altitude KM on the extension of the side AD_1, we have

$$AD_1 = D_1M - AM = \sqrt{KD_1^2 - KM^2} - AM,$$

where M is the foot of the perpendicular from K to AT. Then

$$KD_1 = KB = \frac{3}{2}AB = \frac{3}{2} \cdot 2BL = 3r \sin \frac{\alpha}{2},$$

$$KM = AK \sin KAM = r \sin \frac{\alpha}{2} \sin \frac{\alpha}{2}.$$

Thus

$$KM = r \sin^2 \frac{\alpha}{2},$$

and

$$AM = AK \cos KAM = r \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = \frac{1}{2} r \sin \alpha.$$

Finally,

$$\begin{aligned} AD_1 &= \sqrt{\left(3r \sin \frac{\alpha}{2}\right)^2 - \left(r \sin^2 \frac{\alpha}{2}\right)^2} - \frac{1}{2} r \sin \alpha \\ &= r\left(3 \sin \frac{\alpha}{2} \sqrt{1 - \frac{1}{9} \sin^2 \frac{\alpha}{2}} - \frac{1}{2} \sin \alpha\right). \end{aligned} \tag{12}$$

Now we find an expansion in powers of α for each of the terms in parentheses. First

$$\begin{aligned} \sin \frac{\alpha}{2} &= \frac{\alpha}{2} - \frac{\left(\frac{\alpha}{2}\right)^3}{3!} + \frac{\left(\frac{\alpha}{2}\right)^5}{5!} - \cdots \\ &= \frac{\alpha}{2} - \frac{\alpha^3}{48} + \frac{\alpha^5}{3840} - \cdots. \end{aligned}$$

Next

$$\begin{aligned} \sqrt{1 - \frac{1}{9} \sin^2 \frac{\alpha}{2}} &= \left(1 - \frac{1}{9} \sin^2 \frac{\alpha}{2}\right) \\ &= 1 + \frac{1}{2}\left(-\frac{1}{9} \sin^2 \frac{\alpha}{2}\right) \\ &\qquad + \frac{\frac{1}{2} \cdot \left(\frac{1}{2} - 1\right)}{1 \cdot 2} \cdot \left(-\frac{1}{9} \sin^2 \frac{\alpha}{2}\right)^2 + \cdots \\ &= 1 - \frac{1}{18} \sin^2 \frac{\alpha}{2} - \frac{1}{648} \sin^4 \frac{\alpha}{2} - \cdots. \end{aligned}$$

So far we have a series in powers of $\sin \frac{\alpha}{2}$ $\left(\text{strictly, } \sin^2 \frac{\alpha}{2}\right)$ and not in powers of α. To obtain a power series in α, we could proceed in two ways. First, we could use the method of multiplication of series (ac-

cording to the same rules we used for multiplication of polynomials in Chapter 2)[1]. Then we obtain

$$\sin^2\frac{\alpha}{2} = \left(\frac{\alpha}{2} - \frac{\alpha^3}{48} + \frac{\alpha^5}{3840} - \cdots\right)\left(\frac{\alpha}{2} - \frac{\alpha^3}{48} + \frac{\alpha^5}{3840} - \cdots\right)$$

$$= \frac{\alpha}{2}\cdot\frac{\alpha}{2} - \frac{\alpha}{2}\cdot\frac{\alpha^3}{48} - \frac{\alpha^3}{48}\cdot\frac{\alpha}{2} + \frac{\alpha^5}{3840}\cdot\frac{\alpha}{2}$$

$$+ \frac{\alpha^3}{48}\cdot\frac{\alpha^3}{48} + \frac{\alpha}{2}\cdot\frac{\alpha^5}{3840} - \cdots$$

$$= \frac{\alpha^2}{4} - \frac{\alpha^4}{48} + \frac{\alpha^6}{1440} - \cdots,$$

$$\sin^4\frac{\alpha}{2} = \left(\sin^2\frac{\alpha}{2}\right)^2$$

$$= \left(\frac{\alpha^2}{4} - \frac{\alpha^4}{48} + \frac{\alpha^5}{1440} - \cdots\right)\left(\frac{\alpha^2}{4} - \frac{\alpha^4}{48} + \frac{\alpha^6}{1440} - \cdots\right)$$

$$= \frac{\alpha^2}{4}\cdot\frac{\alpha^2}{4} - \frac{\alpha^4}{48}\cdot\frac{\alpha^2}{4} - \frac{\alpha^2}{4}\cdot\frac{\alpha^4}{48} - \cdots$$

$$= \frac{\alpha^4}{16} - \frac{\alpha^6}{96} + \cdots,$$

etc.

Instead of this, we can obtain the same results more easily with the aid of preliminary trigonometric transformations:

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} = \frac{1 - \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots\right)}{2}$$

$$= \frac{\frac{\alpha^2}{2} - \frac{\alpha^4}{24} + \frac{\alpha^9}{720} - \cdots}{2} = \frac{\alpha^2}{4} - \frac{\alpha^4}{48} + \frac{\alpha^6}{1440} - \cdots,$$

$$\sin^4\frac{\alpha}{2} = \left(\sin^2\frac{\alpha}{2}\right)^2 = \left(\frac{1-\cos\alpha}{2}\right)^2 = \frac{1 - 2\cos\alpha + \cos^2\alpha}{4}$$

$$= \frac{1 - 2\cos\alpha + \frac{1+\cos 2\alpha}{2}}{4}$$

$$= \frac{2 - 4\cos\alpha + 1 + \cos 2\alpha}{8} = \frac{3 - 4\cos\alpha + \cos 2\alpha}{8}$$

[1] In the given case, we first write down the product of all those terms of the series that give the lowest power of α (α^2), then all those that give the next lowest power, α^4, then α^6, etc.

$$= \frac{3 - 4\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots\right)}{8} + \frac{\left[1 - \frac{(2\alpha)^2}{2!} + \frac{(2\alpha)^4}{4!} - \frac{(2\alpha)^6}{6!} + \cdots\right]}{8}$$

$$= \frac{3 - 4 + 2\alpha^2 - \frac{\alpha^4}{6} + \frac{\alpha^6}{180} - \cdots}{8} + \frac{1 - 2\alpha^2 + \frac{4\alpha^4}{6} - \frac{16\alpha^6}{180} + \cdots}{8}$$

$$= \frac{\frac{\alpha^4}{2} - \frac{\alpha^6}{12} + \cdots}{8} = \frac{\alpha^4}{16} - \frac{\alpha^6}{96} + \cdots,$$

etc.

In either case, we find that

$$\sqrt{1 - \frac{1}{9}\sin^2\frac{\alpha}{2}} = 1 - \frac{1}{18}\left(\frac{\alpha^2}{4} - \frac{\alpha^4}{28} + \frac{\alpha^6}{1440} - \cdots\right) - \frac{1}{648}\left(\frac{\alpha^4}{16} - \frac{\alpha^6}{96} + \cdots\right)$$

$$= 1 - \frac{\alpha^2}{72} + \frac{11\alpha^4}{10368} - \cdots.$$

Finally, the last series needed for substitution in the expression for AD_1 is the series for $\sin\alpha$:

$$\sin\alpha = \alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \cdots.$$

Substituting the three series we have found for the corresponding terms in (12), we obtain

$$AD_1 = r\left[3\left(\frac{\alpha}{2} - \frac{\alpha^3}{48} + \frac{\alpha^5}{3840} - \cdots\right)\left(1 - \frac{\alpha^2}{72} + \frac{11\alpha^4}{10368} - \cdots\right) - \frac{1}{2}\left(\alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} - \cdots\right)\right].$$

If we eliminate the parentheses by multiplying the series, we find

$$AD_1 = r\Big[3\cdot\frac{\alpha}{2}\cdot 1 - 3\cdot\frac{\alpha^3}{48}\cdot 1 + 3\cdot\frac{\alpha}{2}\Big(-\frac{\alpha^2}{72}\Big) + 3\cdot\frac{\alpha^5}{3840}\cdot 1$$
$$- 3\cdot\frac{\alpha^3}{48}\Big(-\frac{\alpha^2}{72}\Big) + 3\cdot\frac{\alpha}{2}\cdot\frac{11\alpha^4}{10368} - \cdots$$
$$-\frac{1}{2}\alpha + \frac{1}{2}\cdot\frac{\alpha^3}{6} - \frac{1}{2}\cdot\frac{\alpha^5}{120} + \cdots\Big]$$
$$= r\Big(\frac{3\alpha}{2} - \frac{\alpha^3}{16} - \frac{\alpha^3}{48} + \frac{\alpha^5}{1280} + \frac{\alpha^5}{1152} + \frac{11\alpha^5}{6912} - \cdots$$
$$-\frac{\alpha}{2} + \frac{\alpha^3}{12} - \frac{\alpha^5}{240} + \cdots\Big)$$
$$= r\Big(\alpha - \frac{\alpha^5}{1080} - \cdots\Big) = r\alpha\Big(1 - \frac{\alpha^4}{1080} - \cdots\Big).$$

Thus the difference between the lengths of the arc AB and of the segment AD_1 is

$$\text{arc } AB - \text{length } AD_1 = r\alpha - r\alpha\Big(1 - \frac{\alpha^4}{1080} - \cdots\Big) = \frac{r\alpha^5}{1080} + \cdots.$$

By means of a simple but tedious computation, we can establish that, at least when $\alpha \leq 1$ (57°17′...), we have

$$0 < \text{arc } AB - \text{length } AD_1 < r\frac{\alpha^5}{900}.$$

This method is so accurate that for angles not greater than 30° the absolute error does not reach 0.005% of the radius, and the relative error is less than 0.01% of the measured quantity.

As we see from the last inequality, Rankine's method guarantees six times greater accuracy than Snell's method. For small angles, however, each method gives such an accurate result that the error in the method is completely absorbed by the inaccuracies of the construction (caused by imperfections of the ruler and compass, the fact that the "lines" drawn have width, and similar factors).

7. Logarithms and Series

Logarithms, that invaluable aid to computation, have been in existence for about 350 years, beginning with multiciphered tables for the astronomer or geodesist and ending with the engineer's slide rule. Let the reader imagine that a wonderful time machine has transported him to the beginning of the 17th century, when the ideas of higher mathematics (analytic geometry, differential and integral calculus) only hovered in the minds of a few distinguished mathematicians. Scientists were using essentially only the methods of elementary mathematics. Then the first logarithmic tables (of Bürgi, Briggs, and Napier)[1] appeared.

Could a reader who graduated from a 20th century high school and who knows the advantage and methods of using logarithms join the ranks of "inventors" of logarithms? Could he publish his own table with explanations? Probably many a reader would consider this as impossible as the trip to 300 years ago. The problem of computing the values of logarithms, particularly with great accuracy, appears obscure and complicated. But high school juniors or seniors are capable of compiling a logarithmic table, as we shall see in this chapter.

[1] Joost Bürgi (1552–1632), Swiss, although not a professional scientist (he was a mechanic and a watchmaker), was an able mathematician. However, some of his results were never published and some were published only after a considerable delay (as was the case with his logarithmic tables).

John Napier (1550–1617) was a Scotch mathematician. He not only developed more precise logarithmic tables than Bürgi's, but he also developed the original definition of logarithms, containing some ideas of higher mathematics, which was then only in its beginnings. In addition he is known for his formulas in spherical trigonometry (called Napier's analogies).

Henry Briggs (1561–1630) was an English mathematician and professor at Oxford University. He compiled large (fourteen-digit) tables of logarithms using his own methods of computation.

45. CHANGING THE BASE OF LOGARITHMS

Let us first recall that the logarithm of a positive number N to the positive base a is the exponent of the power to which we must raise a in order to obtain N. In other words, we write

$$\alpha = \log_a N$$

if

$$a^\alpha = N.$$

Thus, in section 35 we saw that

$$1.1^{\sqrt{2}} = 1.14429\ldots;$$

therefore

$$\log_{1.1} 1.14429\ldots = \sqrt{2} = 1.41421\ldots.$$

An important fact is that if we have a logarithmic table computed to any base a, then we are able to compute easily with its aid the values of logarithms to any other base b. For example, as we shall show below, if we multiply the logarithm to base 10 of a number N by 3.3219, we obtain the logarithm of N to base 2; if we multiply its logarithm by 24.160, we obtain its logarithm to base 1.1, and so on. To verify this let us consider the general problem of changing logarithms to one base into logarithms to another base. Let

$$\alpha = \log_a N \quad \text{and} \quad \beta = \log_b N.$$

This means that

$$a^\alpha = N \quad \text{and} \quad b^\beta = N.$$

Hence,

$$a^\alpha = b^\beta.$$

From this relation it is easy to express α in terms of β or β in terms of α. By taking logarithms to base a of both sides of this equation, we obtain

$$\alpha \log_a a = \beta \log_a b,$$

and since $\log_a a = 1$,

$$\alpha = \beta \cdot \log_a b.$$

Replacing α by $\log_a N$ and β by $\log_b N$, we obtain

$$\log_a N = \log_b N \cdot \log_a b. \tag{1}$$

Thus to obtain the logarithm of a number N to a new base a we multiply the logarithm of that number to the old base b by the logarithm of the old base to the new base a.

This formula holds for any value of N. In particular, we can substitute $N = a$ in it to obtain

$$\log_a a = \log_b a \cdot \log_a b,$$

or

$$1 = \log_b a \cdot \log_a b. \tag{2}$$

From this equation it follows that

$$\log_a b = \frac{1}{\log_b a}.$$

If we substitute $\frac{1}{\log_b a}$ for $\log_a b$ in equation (1), we find that

$$\log_a N = \frac{\log_b N}{\log_b a}. \tag{3}$$

This formula gives us the following rule: to find the logarithm of a number to a new base a, we divide the logarithm of that number to the old base b by the logarithm of the new base a to the old base.

We can now verify the statements made earlier. If we wish to find the logarithms of numbers to the base 2, starting with logarithms to the old base 10, then in formula (3) we set $a = 2$, $b = 10$ and find

$$\log_2 N = \frac{\log_{10} N}{\log_{10} 2} = \frac{\log_{10} N}{0.30103} = 3.3219 \log_{10} N.$$

Analogously,

$$\log_{1.1} N = \frac{\log_{10} N}{\log_{10} 1.1} = \frac{\log_{10} N}{0.04139} = 24.160 \log_{10} N,$$

The following rule makes it easier to remember formulas (1), (2), and (3): if in any of these formulas we discard the "log" symbols for logarithms, replace them by fraction bars, and operate on the remaining quantities in the right-hand side as though they were fractions with the bases of logarithms as denominators, then we obtain the left-hand side as the result. Thus, if in formula (1) we discard the logarithm symbols and consider the remaining quantities as numerators and denominators of fractions, we obtain

$$\frac{N}{a} = \frac{N}{b} \cdot \frac{b}{a}.$$

And indeed the left-hand side is equal to the product of the two fractions on the right-hand side.

Analogously, discarding the logarithm symbols in formula (3), we obtain the equation

$$\frac{N}{a} = \frac{\frac{N}{b}}{\frac{a}{b}},$$

in which the left-hand side is again equal to the quotient of the two fractions on the right-hand side. But now we shall return to our problem of compiling a logarithmic table.

46. FINDING A CONVENIENT BASE FOR A LOGARITHMIC TABLE

Since a logarithmic table with one base b can easily be converted to a table with another base a by formula (3), then to compile our first logarithmic table we can use the base for which the computation of logarithms is the easiest. If we later want a table with some other base, we can easily obtain it by multiplying (or dividing) all the logarithms in the table by one number. So, to what base is it easiest to compile a logarithmic table?[1] At first glance, it may appear best to take for this purpose a whole number, 2, 5, 10, 12, etc. But on closer examination a whole number appears inconvenient as a base. Indeed, how would you start to compute with four- or five-digit accuracy

$$\log_{10}7, \quad \log_{10}12, \quad \text{etc.?}$$

On the other hand, how would you begin to solve the reverse problem: finding numbers from their logarithms? For example, what is x if $\log_{10}x = 0.1, 0.27, 0.34$, etc.? Evidently, we could say that

$$x = 10^{0.1} = \sqrt[10]{10} \quad \text{or} \quad x = 10^{0.27} = \sqrt[100]{10^{27}}$$

and similarly, but the actual computation of x would require the computation of roots of various powers of 10, which is again difficult. (We can, of course, use the binomial formula, but at the beginning of the 17th century the general binomial formula was unknown.)

The situation will be different if we select a special number for the base of our system of logarithms. We shall construct a number which can easily be raised to powers with various fractional exponents (a number from which it would be easy to obtain roots with various indices). Such a number is, for example, $b = 2^{10,000}$.

[1] For the time being we shall use only the methods of elementary mathematics for computation of logarithms, as did the first compilers of logarithmic tables (at least Bürgi).

In fact,

$$b^{0.0001} = (2^{10,000})^{0.0001} = 2,$$
$$b^{0.0002} = (2^{10,000})^{0.0002} = 2^2 = 4,$$
$$b^{0.0003} = 2^3 = 8, \quad \text{etc.}$$

Thus we find that the logarithms 0.0001, 0.0002, 0.0003, . . . , etc., to the base b correspond to the numbers 2, 4, 8, ... In this way, we can construct Table III:

TABLE III. Logarithms with $b = 2^{10,000}$

$\log_b N$	N	$\log_b N$	N
0.0000	1	0.0011	2048
0.0001	2	0.0012	4096
0.0002	4	0.0013	8192
0.0003	8	0.0014	16384
0.0004	16	0.0015	32768
0.0005	32	0.0016	65536
0.0006	64	0.0017	131072
0.0007	128	0.0018	262144
0.0008	256	0.0019	524288
0.0009	512	0.0020	1048576
0.0010	1024	. . .	

But this table is almost useless, since the intervals between the numbers N in the right-hand column are too large and, even worse, get rapidly larger and larger. If, for example, we wish to find a number whose logarithm to the base $b = 2^{10,000}$ is equal to 0.00156, then, on the basis of our table, we can only assert that this number lies between 32,768 and 65,536, and this is a wide interval. An error of 0.0001 in the logarithm can cause an error of several hundred thousand or even of millions in the result. The whole trouble, we repeat, is that the fractional powers of the base b increase too rapidly.

To eliminate this defect we shall take, as before, a base b in the form of a power of some number β with a very large exponent. In particular, we shall take $b = \beta^{10,000}$, and we shall choose the number β itself in a special way. The successive fractional powers of b will be

$$b^{0.0001} = (\beta^{10,000})^{0.0001} = \beta, \quad b^{0.0002} = \beta^2,$$
$$b^{0.0003} = \beta^3, \quad b^{0.0004} = \beta^4,$$

For the adjacent powers to be quite close to each other, their ratio should be as close to one as possible. In our case this ratio is

$$\frac{b^{0.0002}}{b^{0.0001}} = b^{0.0001} = \beta,$$

$$\frac{b^{0.0003}}{b^{0.0002}} = b^{0.0001} = \beta,$$

$$\frac{b^{0.0004}}{b^{0.0003}} = b^{0.0001} = \beta, \ldots .$$

Therefore, we should select β as close to one as possible. Thus, taking $\beta = 1.0001$, we obtain as the base of our system of logarithms the number

$$b = (1.0001)^{10,000},$$

which will satisfy our requirements: its fractional powers, whose exponents are multiples of 0.0001 (they represent the integral powers of the number $\beta = 1.0001$), are easily obtained, and the powers for two "adjacent" exponents (differing by 0.0001) differ little from each other. (Their ratio is 1.0001, that is, each successive power is greater than the preceding one by only one ten-thousandth of its magnitude.)

We shall now begin to construct the logarithmic table for the base $b = (1.0001)^{10,000}$. We shall start with the values of the logarithms (that is, exponents of powers to which we raise b) and for these logarithms we shall compute the corresponding numbers N (powers of b). The construction of this table is easy because every successive number in the left-hand column is obtained from the number immediately above it by adding the number 0.0001 (that is, the numbers in the left-hand column form an arithmetic progression with the common difference 0.0001), and every successive number of the right-hand column is obtained from the preceding one by multiplication by the number $\beta = 1.0001$ (that is, the numbers in the right-hand column form a geometric progression with the common ratio 1.0001).

But what does it mean to multiply any number N by 1.0001? This simply means to add to N one ten-thousandth of itself; that is,

$$N \cdot 1.0001 = N + 0.0001 \cdot N.$$

In other words, to multiply some number by 1.0001, we simply move its decimal point four places to the left and add this number to the initial one. We are using just this simple method in constructing Table IV.

TABLE IV. Logarithms with $b = (1.0001)^{10,000}$

$\log_b N$	N	$\log_b N$	N
0.0000	1.000000	0.0101	1.010151
0.0001	1.000100	...	...
0.0002	1.000200	0.0150	1.015112
0.0003	1.000300	0.0151	1.015214
0.0004	1.000400	0.0152	1.015315
0.0005	1.000500	0.0153	1.015417
...	...	...	...
0.0009	1.000900	0.0175	1.017653
0.0010	1.001000	0.0176	1.017755
0.0011	1.001101	—	—
...	...	...	...
0.0050	1.005012	0.0199	1.020098
0.0051	1.005113	0.0200	1.020200
0.0052	1.005213	0.0201	1.020302
0.0053	1.005314	0.0202	1.020404
...	...	...	...
0.0060	1.006018	0.0247	1.025007
...	...	0.0248	1.025110
0.0090	1.009040	0.0249	1.025212
0.0091	1.009141	0.0250	1.025314
...		0.0251	1.025416
—	—	0.0252	1.025519
—	—	0.0253	1.025621
...	...	...	...
0.0099	1.009949	—	—
0.0100	1.010050	—	—

47. LOGARITHMIC TABLES COMPILED IN THE 17TH CENTURY

We observe that one of the first logarithmic tables, Bürgi's table, Table V, differs from our Table IV only in that all the numbers of our left-hand column are increased 100,000 times. A decimal point is to be inferred after the first digit in each number in the body of the table.[1] (Notice also that the numbers are given in the body of the table while the logarithms are in the headings. Thus, $\log_b 1.01714446 = 1700$, where $b = (1.0001)^{10,000}$.)

[1] See *Encyclopaedia Britannica*, article on Logarithms.

Bürgi's tables were published in 1620 under the title *Arithmetic and Geometric Tables of Progressions.* But he was late in publishing his results, which he obtained long before 1620. John Napier, independently from Bürgi, constructed and in 1614 published his logarithmic tables under the title *Description of the Amazing Logarithmic Table.*[1] His table (if we omit the factor 10,000,000 which Napier in-

NAPIER (1550–1617)

troduced to avoid decimals) corresponds to the base $b = \beta^{10,000,000}$ where $\beta = 1 - 0.0000001 = 0.9999999$. Thus the base of Napier's system of logarithms is less than one and therefore the numbers less than one have positive logarithms. Napier wanted to achieve this, as his tables actually give logarithms of sin x and cos x (numbers less than one). In the construction of his tables Napier used essentially the method we explained by constructing logarithms to the base $(1.0001)^{10,000}$.

The methods applied by Napier's friend Briggs in computing logarithms to the base 10 are more diversified. In the year of Napier's death (1617), he published a fourteen-place table of logarithms to the base 10 of numbers up to 1000. Later (1624) in his work *Logarithmic Arithmetic* he gave tables (also fourteen-place) for the numbers from 1 to 20,000 and from 90,000 to 100,000.

[1] The word "logarithm" itself was first used by Napier and formed by him from the Greek words λόγου ἀριθμός meaning "number of a relation". This term, superseded by the term "exponent of the power," was in the spirit of the old mathematical terminology established in ancient Greece..

TABLE V. Bürgi's Logarithms with $B = (1.0001)^{10,000}$

	0	500	1000	1500	2000	2500	3000	3500
0	100000000	100050227	101004966	101511330	102020032	102531384	103045299	103561791
10	····10000	····11277	····15067	····21381	····30234	····41637	····55603	····72140
20	····20001	····21328	····25135	····31524	····40457	····51891	····65909	····82500
30	····30003	····31380	····35271	····41687	····50641	····62146	····76216	····92861
40	····40006	····41433	····45374	····51841	····60846	····72402	····86523	103603221
50	····50010	····51487	····55477	····61996	····71052	····82661	····96837	····13581
60	····60015	····61543	····65584	····71758	····81259	····92918	103717142	····23942
70	····70021	····71599	····75691	····81309	····91467	102603177	····17457	····34305
80	····80028	····81656	····85799	····91460	103001676	····13438	····27764	····44668
90	····90036	····91714	····95907	101602027	····11887	····23699	····38077	····55033
100	100100045	100601773	101105017	····12787	····22098	····33961	····48391	····65398
110	····10055	····11834	····15127	····22649	····32310	····44225	····58705	····75766
120	····10066	····21895	····26239	····33111	····42523	····54489	····69021	····86132
130	····30078	····31952	····36352	····43274	····52738	····64755	····79338	····96501
140	····40091	····42010	····46465	····53438	····62953	····75021	····89656	103706871
150	····50105	····52084	····56580	····63604	····73169	····85289	····99975	····17241
160	····60120	····62150	····66696	····73770	····83386	····95557	103210295	····27613
170	····70136	····72216	····76812	····83938	····93604	103705837	····20616	····37986
180	····80153	····82283	····86930	····94106	102203824	····16097	····30937	····48360
190	····90171	····92351	····97049	101704275	····14045	····26369	····41261	····58734
200	100290190	100792420	101292169	····14446	····24286	····36641	····51585	····69111
210	····10210	····12491	····17289	····24617	····34488	····46915	····61910	····74 47
220	····20231	····22562	····27411	····34790	····44712	····57190	····72237	····8 865
230	····30253	····32634	····37533	····44963	····54936	····67466	····82564	103801244
240	····40276	····42707	····47657	····55136	····65162	····77742	····92897	····10624
250	····50300	····52782	····57782	····65313	····75388	····88020	103303221	····21001
260	····60325	····62857	····67907	····75490	····85616	····98299	····13581	····31387
270	····70351	····72933	····78032	····85668	····95845	102808579	····23883	····41770
280	····80378	····83011	····88162	····95846	102306274	····18860	····34216	····52153
290	····90406	····93089	····98291	101806025	····16307	····29142	····44550	····62564
300	100300435	100803168	101308421	····16206	····26538	····39425	····54883	····72972
310	····10465	····13248	····18552	····26387	····36769	····49708	····65219	····83351
320	····20496	····23330	····28684	····36570	····47003	····59993	····75556	····93737
330	····30528	····33412	····38817	····46754	····57237	····70279	····85893	103904091
340	····40562	····43496	····48950	····56939	····67473	····80566	····96232	····14481
350	····50596	····53580	····59085	····67124	····77710	····90855	103405571	····24873
360	····60631	····63665	····69221	····77311	····87947	102901144	····16912	····35265
370	····70667	····73752	····79358	····87499	····98186	····11434	····27254	····45659
380	····80704	····83839	····89496	····97687	102408426	····21725	····37596	····56057
390	····90742	····93927	····99635	101990787	····18667	····32017	····47941	····66444
400	100400781	100904017	101409775	····18065	····28909	····42310	····58285	····76846
410	····10821	····14107	····19916	····28245	····39152	····52604	····68631	····87243
420	····20862	····24199	····30058	····38425	····49396	····62900	····78971	····97644
430	····30904	····34291	····40201	····48607	····59641	····73196	····89326	104004804
440	····40948	····44384	····50345	····58841	····69887	····83493	····99674	····11444
450	····50991	····54470	····60489	····69037	····80133	····93791	103510024	····22844
460	····61037	····64574	····70636	····79234	····90381	103004091	····20375	····32924
470	····71083	····74671	····80783	····89432	102500630	····14091	····30725	····43005
480	····81131	····84768	····90931	····99631	····10880	····24697	····41081	····53085
490	····91178	····94867	101501080	102009831	····21132	····34991	····51435	····63466
500	100501227	101004966	····11230	····20032	····31384	····45299	····61790	····80816

48. APPROXIMATING THE BASE BY THE BINOMIAL FORMULA

Without dwelling on Briggs's various clever methods (he was an exceptionally talented computer), we shall turn to the computation of logarithms with the aid of infinite series (the simplest method of computation, unknown to Bürgi, Napier, and Briggs). But first we shall consider in more detail bases of the form $b = \beta^n$ where β is quite close to one and n is very large. In our example above (corresponding to the tables of Bürgi), the base was

$$b = (1.0001)^{10,000}.$$

Here the logarithms differed from each other by 0.0001. If we wished to construct in exactly the same way a table where the tabulated logarithms (and the corresponding numbers) differ by smaller values, then we would have to take as a base, for example, the number

$$b_1 = (1.0000001)^{10,000,000}$$

(or

$$b_2 = (0.9999999)^{10,000,000},$$

which would lead us to the tables of Napier). This path would not lead us very far, however. A table in which the logarithms start with 0 and increase by 0.0000001 would contain 10,000,000 pairs of numbers before it reached the logarithm equal to one, which corresponds to the base b_1 itself.

We can represent the bases b and b_1 with the aid of the binomial formula. For example,

$$\begin{aligned} b &= (1.0001)^{10,000} \\ &= (1 + 0.0001)^{10,000} \\ &= 1 + \frac{10,000}{1} 0.0001 + \frac{10,000\,(10,000 - 1)}{1 \cdot 2} (0.0001)^2 \\ &\quad + \frac{10,000\,(10,000 - 1)(10,000 - 2)}{1 \cdot 2 \cdot 3} (0.0001)^3 \\ &\quad + \frac{10,000\,(10,000 - 1)(10,000 - 2)(10,000 - 3)}{1 \cdot 2 \cdot 3 \cdot 4} (0.0001)^4 \\ &\quad + \frac{10,000\,(10,000 - 1)(10,000 - 2)(10,000 - 3)(10,000 - 4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ &\qquad \times (0.00001)^5 + \cdots. \end{aligned}$$

We can make this computation appear less complicated if we transform every term. We observe that the exponent of the number 0.0001 is equal to the number of factors in the numerator of the corresponding binomial coefficient. Therefore, we can rewrite our formula, multiplying each factor in the numerator of the fraction by 0.0001. We obtain

$$
\begin{aligned}
b = 1 &+ \frac{1}{1} \\
&+ \frac{1 \cdot 0.9999}{1 \cdot 2} \\
&+ \frac{1 \cdot 0.9999 \cdot 0.9998}{1 \cdot 2 \cdot 3} \\
&+ \frac{1 \cdot 0.9999 \cdot 0.9998 \cdot 0.9997}{1 \cdot 2 \cdot 3 \cdot 4} \\
&+ \frac{1 \cdot 0.9999 \cdot 0.9998 \cdot 0.9997 \cdot 0.9996}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\
&+ \frac{1 \cdot 0.9999 \cdot 0.9998 \cdot 0.9997 \cdot 0.9996 \cdot 0.9995}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \\
&+ \frac{1 \cdot 0.9999 \cdot 0.9998 \cdot 0.9997 \cdot 0.9996 \cdot 0.9995 \cdot 0.9994}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \\
&+ \frac{1 \cdot 0.9999 \cdot 0.9998 \cdot 0.9997 \cdot 0.9996 \cdot 0.9995 \cdot 0.9994 \cdot 0.9993}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \\
&+ \cdots .
\end{aligned}
$$

If we limit ourselves to the terms we have written (the total number of terms on the right-hand side should be 10,001, as the exponent of the binomial is 10,000) and replace the factors 0.9999, 0.9998, 0.9997, . . . (all quite close to one) in the numerator of each fraction by 1, we obtain the following approximation for the value of b:

$$
\begin{aligned}
b \approx 1 &+ \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\
&+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \\
&+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \cdots .
\end{aligned}
$$

It is easy to prove that the error in this approximate equation does not exceed 0.0002. Therefore with its help we can compute the num-

ber b to within 0.0003, if we change every fraction into a decimal accurate within 0.000005. (For if the error in each of 9 terms does not exceed 0.000005, then the error of the sum does not exceed $9 \cdot 0.000005 = 0.000045$. Adding this error to the error of the formula, at most 0.0002, we obtain $0.000245 < 0.0003$.) We have

$$\begin{aligned}
1 &= 1.00000 \\
\frac{1}{1} &= 1.00000 \\
\frac{1}{1 \cdot 2} &= 0.50000 \\
\frac{1}{1 \cdot 2 \cdot 3} &= 0.16667 \\
\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} &= 0.04167 \\
\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} &= 0.00833 \\
\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} &= 0.00139 \\
\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} &= 0.00020 \\
\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} &= 0.00002 \\
\hline
& 2.71828
\end{aligned}$$

Finally, $b = 2.7183 \pm 0.0003$.

In exactly the same way, we could also compute

$$b_1 = (1.0000001)^{10,000,000}.$$

Only here, after performing transformations similar to those above, the numerators of the fractions differ still less from one and we would make a considerably smaller error than in the preceding case in replacing them by one. It can be stated that to within 0.0000003

$$b_1 = 2.7182818.$$

These two results suggest that the numbers of the form

$$\left(1 + \frac{1}{n}\right)^n$$

(which for $n = 10{,}000$ give b, and for $n = 10{,}000{,}000$ give b_1) tend to a definite limit, approximately equal to 2.71828, as the number n increases indefinitely. Such a limit does in fact exist. This number, which occurs in mathematics almost as frequently as the number π (and, by the way, is connected with π by a simple relation, $e^{\pi i} = -1$), has been designated by the letter e since Euler's time, and is called "Napier's number." Thus

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

49. SERIES EXPANSION FOR e

We can now represent the number e in the form of a series using the binomial formula expansion of $\left(1 + \frac{1}{n}\right)^n$:

$$\left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{1}\left(\frac{1}{n}\right) + \frac{n(n-1)}{1\cdot 2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{n}\right)^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{n}\right)^4 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\left(\frac{1}{n}\right)^5 + \cdots.$$

We can transform this expression in the same way as in the particular case $n = 10{,}000$, obtaining

$$\left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{1} + \frac{1\cdot\left(1 - \frac{1}{n}\right)}{1\cdot 2} + \frac{1\cdot\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{1\cdot 2\cdot 3} + \frac{1\cdot\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right)}{1\cdot 2\cdot 3\cdot 4} + \frac{1\cdot\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right)\left(1 - \frac{4}{n}\right)}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots.$$

(The right-hand side should contain $n + 1$ terms as the expansion of a binomial of the nth degree.) If we now begin to give n rapidly in-

creasing values (for example, $n = 10{,}000$, $n = 100{,}000$, $n = 1{,}000{,}000$, etc.), then the number of terms will increase indefinitely and each of the factors in the numerators will approach one. Thus, the terms on the right-hand side of the formula will individually approach the terms of the infinite series

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \cdots . \tag{4}$$

We encountered this series in Example 4 of section 29 and found that it converges. We could prove (as we did when we derived the series for $\cos x$) that the sum of all the terms of the right-hand side of our formula, that is, the expansion of $\left(1 + \frac{1}{n}\right)$, will approach the sum of the infinite series (4). In other words, the sum of the series (4) represents the limit of $\left(1 + \frac{1}{n}\right)^n$ when n increases indefinitely.[1] Using Euler's notation for this limit, we can write

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \cdots .$$

With the aid of this series, we can compute e with any desired degree of accuracy. Here it is to 19 decimal places:

$$e = 2.7182818284590452353 \ldots .$$

(To obtain this value, it is necessary to take the sum of the first 22 terms of the series, changing each of them into a decimal with accuracy within one-half of the 21st digit after the decimal point.)

[1] Such a proof simultaneously establishes the existence of the limit of $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$ and expresses it in the form of a series.

50. NATURAL LOGARITHMS

When we took the number $b = \left(1 + \frac{1}{10{,}000}\right)^{10{,}000}$ as the base of a logarithmic system, we really took an approximate value of the number e (we have seen that $b = 2.7183 \pm 0.0003$). Therefore, our logarithms also were approximate values of logarithms to the base e. Logarithms to the base e have a special name and notation. We call them the *natural logarithms*[1] and abbreviate them by the symbol

$$\ln N = \log_e N.$$

The letters *ln* in this notation are the first letters of the Latin words *logarithmus naturalis* (natural logarithm). We shall now see how simple it is to expand natural logarithms into a series which we can use to compute these logarithms. (Then, as we have seen, we can compute logarithms to any base.)

51. APPROXIMATE EXPRESSION FOR $\log_b N$

We shall start with logarithms to the base b, where b has the form, $b = \left(1 + \frac{1}{n}\right)^n$ (n is any large number, for example: 10,000; 10,000,000; etc.). If

$$\log_b N = y,$$

then we have

$$b^y = N$$

or

$$\left(1 + \frac{1}{n}\right)^{ny} = N. \tag{5}$$

[1] The name *natural* distinguishes them from the *common* logarithms which have the number 10 as the base. Common logarithms are convenient for calculation because their base, 10, is also the base of our system of numbers. Many other systems of numeration are possible, however: binary (with base 2), ternary (with base 3), $\cdots$, duodecimal (base 12), etc. Some of these are actually used. We find remnants of the sexagesimal system from ancient Babylon in our measurements of time and angles (60 minutes in an hour, 60 seconds in a minute, 60 angular minutes in a degree, 60 seconds in a minute). Modern electronic computers use binary numbers in their calculations. Some languages, for example, French, show traces of a vigesimal system of numeration (with base 20). Our common logarithms would be of little value to a person using a different system of numeration. But the natural logarithms, whose foundation has nothing to do with the system of numeration, would be used just as much and in the same ways in all theoretical investigations.

We shall try to determine y from this equation and in doing so to find a way to compute the logarithm of a given number N. (Previously, when we were constructing a table of logarithms to the base b, we were actually looking for the number corresponding to a given logarithm, not *any* logarithm, but only one of the form: $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n}, \cdots$. In our example, they are 0.0001, 0.0002, 0.0003, 0.0004, etc.)

First we take the nth roots of both sides of equation (5) and then expand the left-hand side into a series by the binomial formula. We obtain

$$\left(1 + \frac{1}{n}\right)^y = N^{\frac{1}{n}},$$

so that

$$1 + \frac{y}{1} \cdot \frac{1}{n} + \frac{y(y-1)}{1 \cdot 2} \frac{1}{n^2} + \frac{y(y-1)(y-2)}{1 \cdot 2 \cdot 3} \frac{1}{n^3} + \cdots = N^{\frac{1}{n}}. \quad (6)$$

We now have an equation in y which, although complicated, is simple to solve approximately. Indeed, if n is a very large number, then $\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}$, etc., are very small numbers. Therefore, if we discard all but the first two terms from the left-hand side of equation (6), we shall make only a small error, which will only slightly affect the quantity y, the desired logarithm. We obtain the approximate equation

$$1 + \frac{y}{n} \approx N^{\frac{1}{n}} \quad (7)$$

with y to the first power only. From this we get

$$y \approx n\left(N^{\frac{1}{n}} - 1\right). \quad (8)$$

To use this formula to find the logarithm of a given number N, we raise N to the power $\frac{1}{n}$. This operation can be carried out with the aid of the binomial series, but only when N has the form $N = 1 + x$ where $|x| < 1$ (for example, $N = 1.736 = 1 + 0.736$, $N = 0.3745 = 1 - 0.6255$, etc.). Under this restriction (later we shall

see how to find logarithms of arbitrary numbers), we obtain

$$y \approx n\left[(1+x)^{\frac{1}{n}} - 1\right] = n\left[1 + \frac{\frac{1}{n}}{1}x + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{1\cdot 2}x^2 + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{1\cdot 2\cdot 3}x^3 + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\left(\frac{1}{n}-3\right)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots - 1\right].$$

If we cancel the $+1$ and -1 inside the brackets and remove the brackets, we have the following expression for y:

$$y \approx \frac{x}{1} + \frac{\frac{1}{n}-1}{1\cdot 2}x^2 + \frac{\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{1\cdot 2\cdot 3}x^3 + \frac{\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\left(\frac{1}{n}-3\right)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots. \tag{9}$$

For very large values of n the number $\frac{1}{n}$ is very small. By discarding the numbers $\frac{1}{n}$ in the numerators of the fractions in formula (9) we cause only an insignificant change in each of these fractions and in their sum (the last assertion requires proof, since this sum contains an infinite number of terms). Granted this, we obtain the following approximate expression for

$$\log_{\left(1+\frac{1}{n}\right)^n}(1+x) = \log_b N = y:$$

$$y \approx \frac{x}{1} + \frac{-1}{1\cdot 2}x^2 + \frac{(-1)(-2)}{1\cdot 2\cdot 3}x^3 + \frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{(-1)(-2)(-3)(-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 + \cdots$$

$$= \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots. \tag{10}$$

Using d'Alembert's test, it is easy to verify that the series in equation (10) is absolutely convergent for any x whose absolute value is less than one.

We have reason to believe that the error in formula (10) becomes smaller as n increases, that is, as the base $b = \left(1 + \frac{1}{n}\right)^n$ of the logarithms approaches Napier's number e, which is the limit of $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$. In the next section we shall actually prove that the approximate equation (10) becomes exact if y denotes a logarithm to the base $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$; that is, if y is the natural logarithm of the number $N = 1 + x$.

52. SERIES EXPANSION FOR ln (1 + x)

To prove that

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots, \quad (11)$$

we need only return to the proof of the approximate formula (10) and evaluate the error in this formula. We began by discarding all but the first two terms in the left-hand side of equation (6); that is, the whole of the infinite series

$$\frac{y(y-1)}{1\cdot 2}\cdot\frac{1}{n^2} + \frac{y(y-1)(y-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3} + \frac{y(y-1)(y-2)(y-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{n^4} + \frac{y(y-1)(y-2)(y-3)(y-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{1}{n^5} + \cdots. \quad (12)$$

To find an estimate for this series we compare its terms with those of a properly chosen geometric series. In the series (12) the ratios of the first few terms to their predecessors are

$$\frac{\dfrac{y(y-1)(y-2)}{1\cdot 2\cdot 3}\cdot\dfrac{1}{n^3}}{\dfrac{y(y-1)}{1\cdot 2}\cdot\dfrac{1}{n^2}} = \frac{y-2}{3}\cdot\frac{1}{n} = \frac{\dfrac{y}{3} - \dfrac{2}{3}}{n},$$

$$\frac{\frac{y(y-1)(y-2)(y-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{n^4}}{\frac{y(y-1)(y-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3}}=\frac{y-3}{4}\cdot\frac{1}{n}$$

$$=\frac{\frac{y}{4}-\frac{3}{4}}{n},$$

$$\frac{\frac{y(y-1)(y-2)(y-3)(y-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{1}{n^5}}{\frac{y(y-1)(y-2)(y-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{n^4}}=\frac{y-4}{5}\cdot\frac{1}{n}$$

$$=\frac{\frac{y}{5}-\frac{4}{5}}{n}.$$

The ratios which follow these are

$$\frac{\frac{y}{6}-\frac{5}{6}}{n},\qquad \frac{\frac{y}{7}-\frac{6}{7}}{n},\qquad \frac{\frac{y}{8}-\frac{7}{8}}{n},\qquad \text{etc.}$$

Obviously, the absolute value of each of these is less than $\frac{|y|+1}{n}$. If we assume that n is so large that $\frac{|y|+1}{n}<\frac{1}{2}$, then in series (12) the absolute value of the ratio of every term to its predecessor will be less than $\frac{1}{2}$. This means that the absolute values of the terms of this series will be no greater than the terms of a geometric progression with first term equal to the absolute value of the first term of the series (12), and common ratio $\frac{1}{2}$. This progression is

$$\frac{|y(y-1)|}{1\cdot 2}\cdot\frac{1}{n^2}+\frac{|y(y-1)|}{1\cdot 2}\cdot\frac{1}{n^2}\cdot\frac{1}{2}+\frac{|y(y-1)|}{1\cdot 2}\cdot\frac{1}{n^2}\left(\frac{1}{2}\right)^2$$
$$+\frac{|y(y-1)|}{1\cdot 2}\cdot\frac{1}{n^2}\left(\frac{1}{2}\right)^3+\frac{|y(y-1)|}{1\cdot 2}\cdot\frac{1}{n^2}\left(\frac{1}{2}\right)^4+\cdots. \qquad (13)$$

Because the absolute values of the terms of the geometric progression (13) are greater than the absolute values of the terms of the series

(12), it follows that the sum of the progression is greater than the absolute value of the sum of the series (12). And since the sum of the progression is

$$\frac{\frac{|y(y-1)|}{1\cdot 2}\cdot\frac{1}{n^2}}{1-\frac{1}{2}}=\frac{\frac{|y(y-1)|}{2}\cdot\frac{1}{n^2}}{\frac{1}{2}}=\frac{|y(y-1)|}{n^2},$$

then

$$\left|\frac{y(y-1)}{1\cdot 2}\cdot\frac{1}{n^2}+\frac{y(y-1)(y-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3}+\frac{y(y-1)(y-2)(y-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{n^4}+\cdots\right|<\frac{|y(y-1)|}{n^2}.$$

In other words, the sum of the series (12) is only a part of the quantity $\frac{y(y-1)}{n^2}$, and we can write

$$\frac{y(y-1)}{1\cdot 2}\cdot\frac{1}{n^2}+\frac{y(y-1)(y-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3}+\frac{y(y-1)(y-2)(y-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{n^4}+\cdots=\alpha\frac{y(y-1)}{n^2},$$

where α is a positive or negative number whose absolute value is less than one; that is, $|\alpha|<1$.

Substituting the expression $\alpha\frac{y(y-1)}{n^2}$ in formula (6), we obtain

$$1+\frac{y}{n}+\alpha\frac{y(y-1)}{n^2}=N^{\frac{1}{n}},$$

from which we have

$$\frac{y}{n}=N^{\frac{1}{n}}-1-\alpha\frac{y(y-1)}{n^2}$$

and

$$y=n\left(N^{\frac{1}{n}}-1\right)-n\alpha\frac{y(y-1)}{n^2}$$

$$=n\left(N^{\frac{1}{n}}-1\right)-\alpha\frac{y(y-1)}{n}.$$

This is no longer an approximation, but an accurate equation, although it has the drawback that the right-hand side contains both the known quantities N and n and the unknowns y and α (about the latter we know only that its absolute value is less than one).

If we again assume that

$$N = 1 + x \quad \text{and} \quad |x| < 1$$

and apply the binomial formula and a simple transformation as before, we find that

$$y = \frac{x}{1} + \frac{\frac{1}{n} - 1}{1 \cdot 2} x^2 + \frac{\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)}{1 \cdot 2 \cdot 3} x^3 + \frac{\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)\left(\frac{1}{n} - 3\right)}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \cdots - \alpha \frac{y(y-1)}{n}.$$

This equation differs from (9) only in that on the right-hand side we have introduced the correction

$$-\alpha \frac{y(y-1)}{n},$$

which makes it accurate.

Now we must evaluate the result of ignoring the small quantities $\frac{1}{n}$ in the numerators of the fractions in the equation. We rewrite the formula, changing the sign of each factor in the numerators of the fractions. This will change the signs of those fractions whose numerators contain an odd number of factors. We obtain

$$y = \frac{x}{1} - \frac{1 - \frac{1}{n}}{1 \cdot 2} x^2 + \frac{\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)}{1 \cdot 2 \cdot 3} x^3 - \frac{\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\left(3 - \frac{1}{n}\right)}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \cdots - \alpha \frac{y(y-1)}{n}. \tag{14}$$

The expressions in the numerators,

$$\left(1-\frac{1}{n}\right), \quad \left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right), \quad \left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\left(3-\frac{1}{n}\right),$$
$$\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\left(3-\frac{1}{n}\right)\left(4-\frac{1}{n}\right), \quad \ldots,$$

can be written as follows:

$$\left.\begin{aligned}
1-\frac{1}{n} &= 1-\frac{1}{n},\\
\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) &= 1\cdot 2-\frac{3}{n}+\frac{1}{n^2}\\
&= 1\cdot 2-\frac{1}{n}\left(3-\frac{1}{n}\right),\\
\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\left(3-\frac{1}{n}\right) &= 1\cdot 2\cdot 3-\frac{11}{n}+\frac{6}{n^2}-\frac{1}{n^3}\\
&= 1\cdot 2\cdot 3-\frac{1}{n}\left(11-\frac{6}{n}+\frac{1}{n^2}\right),\\
\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\left(3-\frac{1}{n}\right)\left(4-\frac{1}{n}\right)&\\
= 1\cdot 2\cdot 3\cdot 4-\frac{50}{n}+\frac{35}{n^2}-\frac{10}{n^3}+\frac{1}{n^4}&\\
= 1\cdot 2\cdot 3\cdot 4-\frac{1}{n}\left(50-\frac{35}{n}+\frac{10}{n^2}-\frac{1}{n^3}\right),&\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&
\end{aligned}\right\} \quad (15)$$

In the right-hand sides of these identities we enclosed in parentheses the terms which contained $\frac{1}{n}$, and factored $\frac{1}{n}$ out. For our purpose we take the absolute value of both sides of each of these identities. Then we substitute in both sides of each of the identities (15) the number -1 for n. Then the identities (15), starting with the second one, become

$$\begin{aligned}
2\cdot 3 &= 1\cdot 2+(3+1),\\
2\cdot 3\cdot 4 &= 1\cdot 2\cdot 3+(11+6+1),\\
2\cdot 3\cdot 4\cdot 5 &= 1\cdot 2\cdot 3\cdot 4+(50+35+10+1),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

These identities can be rewritten in the form

$$2 \cdot 3 - 1 \cdot 2 = (3 + 1),$$
$$2 \cdot 3 \cdot 4 - 1 \cdot 2 \cdot 3 = (11 + 6 + 1),$$
$$2 \cdot 3 \cdot 4 \cdot 5 - 1 \cdot 2 \cdot 3 \cdot 4 = (50 + 35 + 10 + 1),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Having established this, we return to formula (14) and in it we replace the products

$$\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right),$$
$$\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\left(3 - \frac{1}{n}\right),$$
$$\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\left(3 - \frac{1}{n}\right)\left(4 - \frac{1}{n}\right),$$

by their expressions given in the identities (15). We obtain

$$y = \frac{x}{1} - \frac{1 - \frac{1}{n}}{1 \cdot 2} x^2 + \frac{1 \cdot 2 - \frac{1}{n}\left(3 - \frac{1}{n}\right)}{1 \cdot 2 \cdot 3} x^3 - \frac{1 \cdot 2 \cdot 3 - \frac{1}{n}\left(11 - \frac{6}{n} + \frac{1}{n^2}\right)}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \frac{1 \cdot 2 \cdot 3 \cdot 4 - \frac{1}{n}\left(50 - \frac{35}{n} + \frac{10}{n^2} - \frac{1}{n^3}\right)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^5 - \cdots - \alpha \frac{y(y-1)}{n}.$$

Here we break each term (except the first one) into two terms, one not containing the fraction $\frac{1}{n}$ (and its powers) and the other contain-

ing it, and write down first the terms without $\frac{1}{n}$ and then the terms with $\frac{1}{n}$. We obtain

$$y = \frac{x}{1} - \frac{1}{1\cdot 2}x^2 + \frac{1\cdot 2}{1\cdot 2\cdot 3}x^3 - \frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5$$
$$- \cdots + \frac{1}{n}\cdot\frac{1}{1\cdot 2}x^2 - \frac{1}{n}\cdot\frac{3-\frac{1}{n}}{1\cdot 2\cdot 3}x^3$$
$$+ \frac{1}{n}\cdot\frac{11-\frac{6}{n}+\frac{1}{n^2}}{1\cdot 2\cdot 3\cdot 4}x^4 - \frac{1}{n}\cdot\frac{50-\frac{35}{n}+\frac{10}{n^2}-\frac{1}{n^3}}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5$$
$$+ \cdots - \alpha\frac{y(y-1)}{n}. \tag{16}$$

The terms containing $\frac{1}{n}$ as a factor form a series whose sum is easy to estimate when $n > 1$ (in general, we consider only very large positive values of n). Indeed, the absolute value of each term of this series is less than the corresponding term of the geometric progression

$$\frac{1}{n}|x|^2 + \frac{1}{n}|x|^3 + \frac{1}{n}|x|^4 + \frac{1}{n}|x|^5 + \cdots.$$

For the first term, $\frac{1}{n}\cdot\frac{1}{1\cdot 2}x^2$, this is obvious. The absolute value of the second term is

$$\frac{1}{n}\cdot\frac{3-\frac{1}{n}}{1\cdot 2\cdot 3}|x|^3,$$

which is less than

$$\frac{1}{n}\cdot\frac{3+\frac{1}{n}}{1\cdot 2\cdot 3}|x|^3 < \frac{1}{n}\cdot\frac{3+1}{1\cdot 2\cdot 3}|x|^3$$
$$= \frac{1}{n}\cdot\frac{2\cdot 3 - 1\cdot 2}{1\cdot 2\cdot 3}|x|^3 < \frac{1}{n}\cdot\frac{2\cdot 3}{1\cdot 2\cdot 3}|x|^3$$
$$= \frac{|x|^3}{n}.$$

Thus our assertion holds for the second term as well. Analogously, for the third term, we have

$$\frac{1}{n}\cdot\frac{11-\frac{6}{n}+\frac{1}{n^2}}{1\cdot2\cdot3\cdot4}|x|^4<\frac{1}{n}\cdot\frac{11+\frac{6}{n}+\frac{1}{n^2}}{1\cdot2\cdot3\cdot4}|x|^4$$

$$<\frac{1}{n}\cdot\frac{11+6+1}{1\cdot2\cdot3\cdot4}|x|^4$$

$$=\frac{1}{n}\cdot\frac{2\cdot3\cdot4-1\cdot2\cdot3}{1\cdot2\cdot3\cdot4}|x|^4$$

$$<\frac{1}{n}\cdot\frac{2\cdot3\cdot4}{1\cdot2\cdot3\cdot4}|x|^4=\frac{|x|^4}{n}.$$

For the fourth,

$$\frac{1}{n}\cdot\frac{50-\frac{35}{n}+\frac{10}{n^2}-\frac{1}{n^3}}{1\cdot2\cdot3\cdot4\cdot5}|x|^5<\frac{1}{n}\cdot\frac{50+\frac{35}{n}+\frac{10}{n^2}+\frac{1}{n^3}}{1\cdot2\cdot3\cdot4\cdot5}|x|^5$$

$$<\frac{1}{n}\cdot\frac{50+35+10+1}{1\cdot2\cdot3\cdot4\cdot5}|x|^5$$

$$=\frac{1}{n}\cdot\frac{2\cdot3\cdot4\cdot5-1\cdot2\cdot3\cdot4}{1\cdot2\cdot3\cdot4\cdot5}|x|^5$$

$$<\frac{1}{n}\cdot\frac{2\cdot3\cdot4\cdot5}{1\cdot2\cdot3\cdot4\cdot5}|x|^5$$

$$=\frac{|x|^5}{n},$$

etc.

Our assertion holds for all terms containing $\frac{1}{n}$ as a factor in formula (16). Then the sum of all these terms is no greater than the sum of the geometric progression with the common ratio $|x|$ (this is a decreasing progression since $|x|<1$); that is,

$$\frac{1}{n}|x|^2+\frac{1}{n}|x|^3+\frac{1}{n}|x|^4+\frac{1}{n}|x|^5+\cdots=\frac{\frac{|x|^2}{n}}{1-|x|}$$

$$=\frac{|x|^2}{n(1-|x|)}.$$

Hence, we can assert that

$$\frac{1}{n}\cdot\frac{1}{1\cdot 2}x^2 - \frac{1}{n}\frac{3-\frac{1}{n}}{1\cdot 2\cdot 3}x^3 + \frac{1}{n}\cdot\frac{11-\frac{6}{n}+\frac{1}{n^2}}{1\cdot 2\cdot 3\cdot 4}x^4 - \frac{1}{n}\cdot\frac{50-\frac{35}{n}+\frac{10}{n^2}-\frac{1}{n^3}}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 + \cdots = \beta\frac{|x|^2}{n(1-|x|)},$$

where β is a positive or negative number whose absolute value is less than one ($|\beta| < 1$). Thus from formula (16), instead of the approximate formula (10), we obtain a new formula

$$y = \log_{\left(1+\frac{1}{n}\right)^n}(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + \beta\frac{|x|^2}{n(1-|x|)} - \alpha\frac{y(y-1)}{n}. \qquad (17)$$

This formula allows us to estimate the error in formula (10). This error is

$$\beta\frac{|x|^2}{n(1-|x|)} - \alpha\frac{y(y-1)}{n}.$$

We notice that for very large n this error is very small if $|x|$ is not close to one. Thus, when the base of the logarithms is $b = \left(1+\frac{1}{10{,}000}\right)^{10{,}000}$ (the case $n = 10{,}000$), the error in formula (10) is less than

$$\frac{|x|^2}{10{,}000\,(1-|x|)} + \frac{|y(y-1)|}{10{,}000}.$$

(Recall that $|\alpha|$ and $|\beta|$ are smaller than one.) If $|x| \le \frac{1}{2}$ then $|y| = \log_b(1+x)$ must be less than one. In other words, $\frac{1}{2} \le 1 + x \le 1\frac{1}{2}$, and since $b = 2.718\ldots$ (p. 164),

$$\log_b 1\tfrac{1}{2} < \log_b 2.718\ldots = 1$$

and

$$\left|\log_b \tfrac{1}{2}\right| = |\log_b 1 - \log_b 2| = |-\log_b 2| < \log_b 2.718\ldots = 1.$$

In this case, the error in formula (10) is less than

$$\frac{\frac{1}{4}}{10{,}000 \cdot \frac{1}{2}} + \frac{1 \cdot 2}{10{,}000} = 0.00025.$$

But if x is equal, for example, to 0.999, then the term $\frac{|x|^2}{10{,}000\,(1 - |x|)}$ is approximately equal to $\frac{1}{10{,}000\,(1 - 0.999)} = \frac{1}{10} = 0.1$. Hence, the error resulting from the use of the formula (10) may not be small.

To complete the proof of formula (11), we take limits in both parts of formula (17) as $n \to \infty$. As we know by formula (3) of this chapter,

$$\log_{\left(1 + \frac{1}{n}\right)^n}(1 + x) = \frac{\log_b (1 + x)}{\log_b \left[\left(1 + \frac{1}{n}\right)^n\right]} = \frac{\ln (1 + x)}{\ln \left[\left(1 + \frac{1}{n}\right)^n\right]}.$$

As $n \to \infty$ the quantity $\left(1 + \frac{1}{n}\right)^n$ approaches e and therefore[1]

$$\ln \left[\left(1 + \frac{1}{n}\right)^n\right] \to \ln e = 1$$

and

$$\log_{\left(1 + \frac{1}{n}\right)^n}(1 + x) \to \frac{\ln (1 + x)}{1} = \ln (1 + x).$$

On the right-hand side of formula (17), we see that the terms $\beta \frac{|x|^2}{n(1 - |x|)}$ and $\alpha \frac{y(y - 1)}{n}$ approach zero as $n \to \infty$ and that the remaining terms do not depend on n.

Now we have proved formula (11):

$$y = \ln (1 + x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots,$$

the basic formula of this chapter.

[1] We can prove this as follows. First we observe that

$$\left(1 + \frac{1}{n}\right)^n < e. \qquad (*)$$

This follows from the term-by-term comparison of these expressions for $\left(1 + \frac{1}{n}\right)^n$ (see p. 165):

$$\left(1+\frac{1}{n}\right)^n = 1+\frac{1}{1}+\frac{1\left(1-\frac{1}{n}\right)}{1\cdot 2}+\frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\cdot 2\cdot 3}+\cdots$$
$$+\frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)}{1\cdot 2\cdot 3\cdots n},$$

and for e (p. 166):

$$e = 1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots+\frac{1}{1\cdot 2\cdot 3\cdot\ldots\cdot n}+\cdots.$$

Taking the natural logarithms of both sides of (*), we obtain

$$\ln\left[\left(1+\frac{1}{n}\right)^n\right] < \ln e,$$

that is,

$$n\ln\left(1+\frac{1}{n}\right) < 1,$$

or finally

$$\ln\left(1+\frac{1}{n}\right) < \frac{1}{n}. \qquad (**)$$

We have to prove that $\ln\left[\left(1+\frac{1}{n}\right)^n\right] \to \ln e$; that is, that $\ln e - \ln\left[\left(1+\frac{1}{n}\right)^n\right] \to 0$ as $n\to\infty$. But the difference of two logarithms is the logarithm of the quotient:

$$\ln e - \ln\left[\left(1+\frac{1}{n}\right)^n\right] = \ln\left[\frac{e}{\left(1+\frac{1}{n}\right)^n}\right].$$

Since $\left(1+\frac{1}{n}\right)^n \to e$ as $n\to\infty$, the number $\frac{e}{\left(1+\frac{1}{n}\right)^n}$, which exceeds 1 by virtue of inequality (*), approaches 1 as $n\to\infty$. Therefore we can write

$$\frac{e}{\left(1+\frac{1}{n}\right)^n} = 1+\varepsilon_n,$$

where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n\to\infty$.

Now we must prove that

$$\ln(1+\varepsilon_n) \to 0 \text{ as } \varepsilon_n \to 0.$$

For any natural number N, the positive number ε_n will be less than $\frac{1}{N}$ for sufficiently large values of the subscript n. Then $\ln(1+\varepsilon_n)$ will be less than $\ln\left(1+\frac{1}{N}\right)$ which, in turn, by inequality (**), will be less than $\frac{1}{N}$. We have established that for sufficiently large n

$$0 < \ln e - \ln\left[\left(1+\frac{1}{n}\right)^n\right] = \ln(1+\varepsilon_n) < \frac{1}{N}$$

for any N given beforehand. From this it follows that

$$\ln e - \ln\left[\left(1+\frac{1}{n}\right)^n\right] \to 0 \text{ as } n\to\infty$$

which means that our passage to the limit is well-grounded.

53. COMPUTATION OF NATURAL LOGARITHMS

With formula (11), we can compute only the natural logarithms of numbers of the form $N = 1 + x$ where $|x| < 1$, because we derived the formula under this assumption. (If we took $|x| > 1$ then we could not use the binomial series at all.) It is possible, however, to prove that formula (11) also holds for $x = 1$. In other words,

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots . \tag{18}$$

But this series converges very slowly and it is necessary to take a very great number of terms in order to obtain ln 2 with any significant accuracy. For example, to compute a result accurate within 0.001, it is necessary for the first of the discarded terms of the series to be less than 0.001. But this will mean the 1001st term, so that computing ln 2 by formula (18) to an accuracy of 0.001 requires taking the sum of 1000 terms of the series. We can easily avoid this difficulty by computing $\ln \frac{1}{2}$ instead of ln 2 by formula (11), as follows:

$$\begin{aligned}
\ln 2 &= -\ln \frac{1}{2} \\
&= -\ln\left(1 - \frac{1}{2}\right) \\
&= -\left[\frac{\left(-\frac{1}{2}\right)}{1} - \frac{\left(-\frac{1}{2}\right)^2}{2} + \frac{\left(-\frac{1}{2}\right)^3}{3}\right. \\
&\qquad - \frac{\left(-\frac{1}{2}\right)^4}{4} + \frac{\left(-\frac{1}{2}\right)^5}{5} \\
&\qquad \left. - \frac{\left(-\frac{1}{2}\right)^6}{6} + \frac{\left(-\frac{1}{2}\right)^7}{7} - \cdots\right] \\
&= \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} \\
&\qquad + \frac{1}{5 \cdot 2^5} + \frac{1}{6 \cdot 2^6} + \frac{1}{7 \cdot 2^7} + \cdots .
\end{aligned}$$

To compute ln 2 to an accuracy of 0.001, it is now sufficient to compute the sum of the first seven terms:

$$\begin{aligned}
\frac{1}{2} &= 0.50000 \\
\frac{1}{2 \cdot 2^2} = \frac{1}{8} &= 0.12500 \\
\frac{1}{3 \cdot 2^3} = \frac{1}{24} &= 0.04167 \\
\frac{1}{4 \cdot 2^4} = \frac{1}{64} &= 0.01562 \\
\frac{1}{5 \cdot 2^5} = \frac{1}{160} &= 0.00625 \\
\frac{1}{6 \cdot 2^6} = \frac{1}{384} &= 0.00260 \\
\frac{1}{7 \cdot 2^7} = \frac{1}{896} &= 0.00112 \\
&\overline{0.69226.}
\end{aligned}$$

We find that $\ln 2 = 0.692 \pm 0.001$.

From formula (11) we can obtain series that will give us logarithms of arbitrary numbers, as well as series that converge much more rapidly than series (11). To do this we first replace x by $-x$ in formula (11). We obtain

$$\ln (1 - x) = \frac{(-x)}{1} - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \cdots$$

or

$$\ln (1 - x) = -\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots . \qquad (19)$$

Now we subtract equation (19) from equation (11) term by term to find

$$\ln (1 + x) - \ln (1 - x) = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots ,$$

or

$$\ln \frac{1 + x}{1 - x} = 2\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots\right), \qquad (20)$$

which is valid for $|x| < 1$. In particular, setting $x = \dfrac{1}{2N + 1}$ in (20),

where N is a positive integer, we find

$$\ln \frac{1 + \frac{1}{2N+1}}{1 - \frac{1}{2N+1}} = 2\left[\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots\right]$$

or, observing that

$$\frac{1 + \frac{1}{2N+1}}{1 - \frac{1}{2N+1}} = \frac{2N+1+1}{2N+1-1} = \frac{2N+2}{2N} = \frac{N+1}{N},$$

and that

$$\ln \frac{1 + \frac{1}{2N+1}}{1 - \frac{1}{2N+1}} = \ln \frac{N+1}{N} = \ln (N+1) - \ln N,$$

we finally obtain the formula

$$\ln (N+1) = \ln N + 2\left[\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots\right]. \quad (21)$$

This formula allows us to find $\ln (N+1)$ if we know $\ln N$.

For example, knowing that $\ln 1 = 0$ and setting $N = 1$ in formula (21), we can assert that

$$\ln 2 = 2\left[\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \cdots\right].$$

Computing the sum of the first four terms of the series, we obtain $\ln 2$ accurate to 0.00002:

$$\ln 2 = 0.69313.$$

The sum of the first eight terms of the series gives the result

$$\ln 2 = 0.69314718$$

with an error less than 0.00000001.

We shall compute with the aid of series (21) the natural logarithm of 10. Since

$$\ln 10 = \ln 2 + \ln 5$$

we only have to find the logarithm of 5. Formula (21) allows us to find logarithms of the number $(N + 1)$ only when we know the logarithm of N. In this case, to find ln 5 we have to know ln 4. But the latter is

$$\ln 4 = 2 \cdot \ln 2 = 2 \cdot 0.69314718 = 1.38629436.$$

Setting $N = 4$ in formula (21), we find

$$\ln 5 = \ln 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7} + \cdots\right)$$

or, computing the sum of the terms shown and substituting the value of ln 4, we obtain

$$\ln 5 = 1.38629436 + 2 \cdot 0.111571775 = 1.60943791.$$

Consequently,

$$\ln 10 = \ln 2 + \ln 5 = 0.69314718 + 1.60943791 = 2.30258509,$$

a result accurate up to 0.00000003.

54. COMPUTATION OF LOGARITHMS TO THE BASE 10

Now we can derive a formula for the computation of logarithms to the base 10. For this we need only recall formula (3) of this chapter (p. 155), which in this case allows us to shift from natural to common logarithms. By it we have

$$\log_{10} N = \frac{\ln N}{\ln 10} = \frac{\ln N}{2.30258509} = 0.43429448 \cdot \ln N. \qquad (22)$$

Thus, to find the common logarithm of a number N when we know its natural logarithm, we multiply the natural logarithm by 0.43429448. We usually denote this number by the letter M

$$M = 0.43429448$$

and call it the *modulus* of the system of common logarithms. When our computations do not require great accuracy, we can use this easy-to-remember value for M:

$$M \approx 0.4343.$$

Now we can derive two formulas for the computation of common logarithms from our two basic known formulas for computation of natural logarithms [(11) and (21)]:

$$\log_{10}(1+x) = 0.43429448\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) \tag{23}$$

and

$$\log_{10}(N+1) = \log_{10} N + 2 \cdot 0.43429448\left[\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots\right]$$

or

$$\log_{10}(N+1) = \log_{10} N + 0.86858896\left[\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots\right]. \tag{24}$$

We shall give some of the details of the construction of logarithmic tables from formula (24). Let us suppose that we have to construct a five-place logarithmic table; that is, we have to compute logarithms accurate to within 0.000005.

If we have the logarithms of all four-digit numbers (1000, 1001, 1002, . . . , 9999), we can find the logarithms of arbitrary numbers up to five significant digits even if they are not contained in the interval from 1000 to 10,000. By a simple rule and without a table, we can find the integral part (the characteristic) of a common logarithm. The fractional part (the *mantissa*) of a common logarithm does not depend upon the location of the decimal point but only on the digits in the number and on their order. These two facts, which can be explained by our decimal system of numbers, are the basic advantages of the system of common logarithms over all other systems.

For instance, in finding the logarithm of 11.506, we know that the characteristic of the logarithm is 1 and the mantissa is the same as in the logarithm of 1150.6 (this number lies between 1,000 and 10,000). But in our table there will be no mantissas of the logarithms of numbers between 1150 and 1151. We find the correction, corresponding to the last digit (6) of the number 1150.6, by the method of linear interpolation.

If we take only one term inside the brackets in formula (24), we obtain formula (25), which we can use to compute the logarithm of the number $N + 1$ from the known logarithm of the number N:

$$\log (N + 1) \approx \log N + \frac{0.86858896}{2N + 1}. \tag{25}$$

Thus, knowing that $\log 1000 = 3$ we can find $\log 1001$:

$$\begin{aligned}\log 1001 &\approx \log 1000 + \frac{0.86858896}{2001} \\ &= 3 + 0.0004340769 = 3.0004340769.\end{aligned}$$

Then we can obtain $\log 1002$:

$$\begin{aligned}\log 1002 &\approx \log 1001 + \frac{0.86858896}{2003} \\ &= 3.0004340769 + 0.0004336435 = 3.0008677204,\end{aligned}$$

then $\log 1003$, $\log 1004$, $\log 1005, \ldots$. Note the accuracy of formula (25) we are using. We obtained it from formula (24) by discarding the terms

$$0.86858896\left[\frac{1}{3(2N + 1)^3} + \frac{1}{5(2N + 1)^5} + \frac{1}{7(2N + 1)^7} + \frac{1}{9(2N + 1)^9} + \cdots\right].$$

If we replace the factor 0.86858896 by the larger number 1 and the numbers 5, 7, 9, ... in the denominators of the fractions by the smaller number 3, we must obtain a larger sum,

$$\frac{1}{3(2N + 1)^3} + \frac{1}{3(2N + 1)^5} + \frac{1}{3(2N + 1)^7} + \cdots,$$

which, as the sum of an infinite decreasing geometric progression with common ratio $\left(\frac{1}{2N + 1}\right)^2$, is equal to

$$\frac{\frac{1}{3(2N + 1)^3}}{1 - \frac{1}{(2N + 1)^2}}.$$

Thus the error in formula (25) is less than

$$\frac{1}{3(2N+1)^3} \cdot \frac{1}{1-\frac{1}{(2N+1)^2}},$$

and for $N > 1000$ is less than

$$\frac{1}{3 \cdot 2001^3} \cdot \frac{1}{1-\frac{1}{2001^2}} < 0.00000000005,$$

that is, less than half of the tenth digit after the decimal point. Such accuracy is guaranteed, however, only when $\log N$, which appears in the right-hand side of formula (25), is exact (contains no error). Of course, we cannot expect this from our method. Indeed, $\log 1000$ is exactly 3 and therefore the error in the determination of $\log 1001$ by formula (25) is no greater than 0.00000000005. But in computing $\log 1002$ we start with an approximate value of $\log 1001$ (accurate up to one-half of the tenth digit after the decimal point). To this we must again add the error of the formula, that is, another 0.00000000005. The whole error in determining $\log 1002$ will be on the order of $2 \cdot 0.00000000005 = 0.0000000001$. Continuing, we find that in the determination of $\log 1003$ the error may reach $3 \cdot 0.00000000005$, in the determination of $\log 1004$ the error may reach $4 \cdot 0.00000000005$, etc. In this way, at the end of the table in the determination of $\log 9999$ the error may reach $8999 \cdot 0.00000000005 \approx 0.0000005$.

Also, we did not consider the errors in transforming the fraction $\frac{0.86858896}{2N+1}$ into a decimal. If we compute this quantity to ten decimal places (accurate to within 0.00000000005), then the error in the formula for $\log (N+1)$ (in spite of the accurate value of $\log N$) will be $2 \cdot 0.00000000005 = 0.0000000001$. At the end of the table this error will be 0.000001, which is twice as great as the one computed above. Consequently, although we carry out the computations by formula (25) to ten decimal places we can guarantee only six correct digits after the decimal point in the logarithms at the end of the table.

This is sufficient for the accuracy we wanted, but it is inconvenient to carry out all our computations with ten-digit numbers in order to obtain results with only six accurate digits. To avoid the unnecessary work, we can compute the logarithms of the two-digit numbers 10, 11, . . . , 98, 99 with six accurate digits. We do this with

the aid of formula (24), where we use enough of the terms within the brackets to obtain logarithms accurate to within 0.000001 (at first we need two terms of the formula, later only one).

Having these logarithms, we shall immediately know with the same accuracy the logarithms of the numbers 1000, 1100, 1200, 1300, 1400, . . . , 9800, 9900, that is, the logarithms of the four-digit multiples of 100. Then we again use formula (25) to compute the logarithms of the remaining four-digit numbers, but this time we do not let the error accumulate over the whole range from 1000 to 10,000, but we use "relays" after every hundred. We find by formula (25) the logarithms of 1001, 1002, 1002, 1004, 1005, . . . , and when we reach 1099 we do not compute log 1100 by formula (25) but instead we use the previously computed value of this logarithm. Now we again compute by formula (25) the logarithms of 1101, 1102, 1103, . . . , and the error again accumulates. We do not let it become great, however, because after reaching log 1199 we do not compute log 1200 by formula (25) but take its previously computed value.

We proceed in the same way to the end of the table. Now the error accumulates not over a stretch of 9000 operations (from 1000 to 10,000) but each time over a stretch of only 100 operations (from 1000 to 1100, from 1100 to 1200, from 1200 to 1300, etc.). Therefore, carrying out all the computations by formula (25) to eight decimal places (accurate up to 0.000000005), we never accumulate an error greater than

$$2 \cdot 0.000000005 \cdot 100 = 0.000001.$$

In addition, errors not greater than 0.000001 are already present in the logarithms of the numbers 1100, 1200, 1300, 1400, . . . (that is, in the logarithms of the numbers with which we start each 100 operations). Hence, the values of all the resulting logarithms will be accurate to within 0.000002, which is more accurate than we needed in our five-place table.